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TABLE OF CONTENTS

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Mathematical Reviews

Vol. 9, No. 4 APRIL, 1948 Pages 169-220

HISTORY

Sachs, A. J. Babylonian mathematical texts. I. Reciprocals of regular sexagesimal numbers. J. Cuneiform Studies 1, 219–240 (1947).

If we call an integer n regular if it contains no prime numbers other than 2, 3 and 5 then all and only regular numbers have reciprocals $\bar{n}$ whose expansion in a sexagesimal fraction is finite. It has been shown that the Babylonians used a certain list of pairs n and $\bar{n}$ as a standard table. The author is now able to show the procedure which was followed in order to find the reciprocal $\bar{c}$ of a regular number c if neither c nor $\bar{c}$ occurs in the standard table. The method consists in splitting c into two parts a and b such that $\bar{a}$ is a number of the standard table and then using the identity $\bar{c}=\bar{a}(\overline{1+b\bar{a}})$, iterating this process, if necessary, until the second factor also appears in the standard table. A whole group of published and unpublished texts can now be understood as applying this technique.
O. Neugebauer (Providence, R. I.).

Lewy, Hildegard. Marginal notes on a recent volume of Babylonian mathematical texts. J. Amer. Oriental Soc. 67, 305–320 (1947).

Remarks concerning some of the problem-texts published recently by Neugebauer and Sachs [Mathematical Cuneiform Texts, New Haven, 1945; these Rev. 8, 1].
O. Neugebauer (Providence, R. I.).

van der Waerden, B. L. Egyptian "Eternal tables." II. Nederl. Akad. Wetensch., Proc. 50, 782–788 (1947).

This is the concluding part of the author's investigation of Egyptian planetary texts [same vol., 536–547; these Rev. 9, 73], now concerning the motion of Saturn and Mercury. Qualitative agreement with Babylonian methods is established but accurate numerical coincidences are lacking. Finally the reduction of lunar dates is discussed; the author goes so far as to assume the use of the Babylonian calendar for the original computation. *O. Neugebauer.*

Mogenet, J. Les définitions dans l'ancienne sphérique. Ann. Soc. Sci. Bruxelles. Sér. I. 61, 235–241 (1947).

The author investigates the definitions in the spherics of Autolycus, Euclid and Theodosius in order to reconstruct the definitions in the original version. *O. Neugebauer.*

Shankar Shukla, Kripa. The evection and the deficit of the equation of the centre of the moon in Hindu astronomy. Proc. Benares Math. Soc. (N.S.) 7, no. 2, 9–28 (1945).

The author shows that as early as A.D. 900 formulae were known which gave periodic perturbations for the longitude and the radius vector of the moon. Because these perturbations have as their argument the elongation and the distance of the sun from the perigee of the moon it is obvious that they reflect the inequality known as evection and discovered by Ptolemy. Ptolemy describes this phenomenon by means of the movement of the deferent of the lunar epicycle on an eccenter combined with a periodic displacement of the apogee of the epicycle. The model of the Hindu astronomers consists of a displacement of the earth depending on the above mentioned variables. Both theories give very satisfactory results for the longitudes. The Hindu theory seems to be much better as far as distances are concerned. *O. Neugebauer* (Providence, R. I.).

***Carmody, Francis J. Leopold of Austria: "Li Compilacions de le Science des Estoilles," Books I–III.** Edited from MS French 613 of the Bibliothèque Nationale, with notes and glossary. University of California Publications in Modern Philology 33, no. 2, pp. i–iv+35–102 (1947).

The "Compilatio" is of interest for the transmission of Arabic and Hindu astronomy to western Europe. The original Latin version was probably written in 1271. This is an edition of the first three books of the French translation, prepared before 1324. Especially the second book is of interest because of its explanation of the theory of the lunar and planetary movement. The editor gives a sketch of the history of the Compilatio, postponing a detailed investigation of its sources. *O. Neugebauer* (Providence, R. I.).

Boyer, Carl B. Note on epicycles and the ellipse from Copernicus to Lahire. Isis 38, 54–56 (1947).

Tibiletti, Cesarina. Sul problema di Apollonio: i cerchi orientati e le soluzioni di Vieta, Plücker e Newton. Period. Mat. (4) 25, 16–29 (1947).

Conte, Luigi. Sul modo di mettere in equazione le questioni geometriche. (Dall'"Arithmetica Universalis" di I. Newton.) Period. Mat. (4) 25, 1–15 (1947).

***Dessauer, Friedrich. Weltfahrt der Erkenntnis, Leben und Werk Isaac Newtons.** Rascher Verlag, Zürich, 1945. 430 pp. (8 plates). 14.5 Swiss francs unbound; 17.5 Swiss francs bound.

Dieses Buch ist keine Newton-Biographie im traditionellen Sinn. Um dem Leser einen lebendigen Einblick zu verschaffen in das Gefühls- und Gedankenleben des grossen englischen Forschers und ihm die Überzeugung von der welthistorischen Bedeutung seines Schaffens beizubringen, hat der Verf. eine künstlerische Darstellungsform bevorzugt. Es werden Gespräche geführt, an denen Newton selbst und seine Verwandten, ferner zwei (fingierte) Jugendfreunde und einige bedeutende Gelehrten seines Zeitalters wie Barrow, Locke und Wallis teilnehmen und in denen seine optischen Entdeckungen, die Fluxionsrechnung, der Aufbau der Mechanik und deren Anwendung in der Astronomie eingehend diskutiert werden. In der nämlichen Weise wird die historisch-politische Lage geschildert, die den Hintergrund zu Newton's Leben abgibt, eine Erklärung gegeben für die körperliche und seelische Gesundheitskrise, die sein Schaffen eine Zeit lang gehemmt hat und der seelische Konflikt

aufgezeigt, den die Diskrepanz zwischen den religiösen Absichten, die ihn bei seiner wissenschaftlichen Produktion geleitet hatten und den so ganz anders gearteten weltanschaulichen Einflüssen, die seine Theorieen schon zur Zeit seines Lebens ausübten, auslöste. Der Verf. illustriert in dieser Weise seine These von dem selbstständigen Eigenleben, das eine Erkenntnis, indem sie sich von ihrem Urheber löst, antritt und weist auf die grossen geistigen Gefahren hin, die entstehen wenn eine an sich richtige naturwissenschaftliche Betrachtungsweise, die aber, ihrer Art gemäss, immer nur eine partielle Bedeutung haben kann, zu einer allgemeinen metaphysichen Lehre verallgemeinert wird. In diesem Zusammenhang wird die relative Berechtigung der bei den Naturforschern des XVII. Jahrhunderts absolut verpönten thomistischen Naturphilosophie hervorgehoben. In zwei Reihen von Ergänzungen und Erläuterungen wird wissenschaftliche Rechnung abgelegt über das im Buche in künstlerischer Form dargestellte. *E. J. Dijksterhuis.*

***Lambert, Johann Heinrich. Schriften zur Perspektive. Herausgegeben und eingeleitet von Max Steck.** Dr. Georg Lüttke Verlag, Berlin, 1943. xvii+496 pp. (21 plates)

The volume includes a previously unpublished manuscript, "Anlage zur Perspektive" [1752], and bibliographies of works by and concerning Lambert.

Burckhardt, Johann Jakob. Der mathematische Nachlass von Ludwig Schläfli (1814–1895) an der Schweizerischen Landesbibliothek. Mitt. Naturforsch. Ges. Bern 1942, 1–22 (1943).

Stewart, C. A. Obituary: P. J. Daniell. J. London Math. Soc. 22, 75–80 (1947).

Gomes, Ruy Luís. Obituary: T. Levi-Civita. Anais Fac. Ci. Pôrto 28, 5–7 (1 plate) (1943). (Portuguese)

ALGEBRA

* This sextic resolvent of a quintic equation was computed earlier by W. E. H. Berwick. [Proc. London Math. Soc. (2) 14, 301–307 (1915)]

Berzolari, Luigi. Sopra una risolvente, a coefficienti invarianti, dell'equazione di quinto grado. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 8(77), 574–578 (1944).

A resolvent of sixth degree of the general quintic equation is constructed. The coefficients are invariants. Necessary and sufficient conditions for the solubility of a quintic equation by means of radicals can be derived. *R. Brauer.*

Skolem, Th. The non-symmetric functions in algebra. Norsk Mat. Tidsskr. 29, 65–74 (1947). (Norwegian)

A discussion of the application of permutation groups to rational functions in several variables. *O. Ore.*

Segre, Beniamino. Equivalenza ed automorfismi delle forme binarie in un dato anello o campo numerico. Univ. Nac. Tucumán. Revista A. 5, 7–68 (1946).

The author has previously [Proc. Cambridge Philos. Soc. 41, 187–209 (1945); these Rev. 7, 169] given necessary and sufficient conditions that two binary n-ics f and f_1, with nonvanishing discriminants, are equivalent over the complex field C; that is, that there exist complex numbers α, β, γ, δ such that $f_1(x, y)=f(\alpha x+\beta y, \gamma x+\delta y)$. In the first part of the present paper he shows how these conditions can be applied to determine the equivalence of binary forms over subfields or subrings R of C, and applies them to the cases $n=3$ and 4, R being a real field. For $n=2$ a criterion is given which holds for any field of characteristic different from 2.

The rest of the paper is devoted to the automorphisms of a binary n-ic over the rational field. The form f is weakly automorphic if there exist $\alpha, \beta, \gamma, \delta$ other than $\alpha, 0, 0, 0$ such that $f(\alpha x+\beta y, \gamma x+\delta y)=\epsilon f(x, y)$, $\epsilon\neq 0$; f is strongly automorphic if $\epsilon=1$. The automorphisms of a given n-ic, $n>2$, constitute a finite group; the author's first step is to determine all finite groups of rational one-dimensional projectivities, $x_1=(\alpha x+\beta)/(\alpha x+\delta)$ and of rational two-dimensional substitutions $x_1=\alpha x+\beta y$, $y_1=\gamma x+\delta y$. Excluding a few trivial cases there exist eight of each of these, cyclic groups of orders 2, 3, 4, and 6, and dihedral groups of orders 4, 6, 8, and 12. The structure of a binary n-ic invariant under each of these groups is then determined and the cases $n=2$, 3 and 4 are investigated in detail. *R. J. Walker.*

Ferrar, W. L. The simultaneous reduction of two real quadratic forms. Quart. J. Math., Oxford Ser. 18, 186–192 (1947).

The simultaneous real reduction of two real quadratic forms $A(x, x)$ and $C(x, x)$, the latter positive-definite, to the forms $\sum\lambda_r X_r^2$ and $\sum X_r^2$ by a method which is "free of difficult invariant-factor arguments." *C. C. MacDuffee.*

Todd, J. A. A note on real quadratic forms. Quart. J. Math., Oxford Ser. 18, 183–185 (1947).

If A is a real symmetric $n\times n$ matrix and if λ_1 is a p-fold root of $|A-\lambda I|=0$, then $A-\lambda_1 I$ is of rank $n-p$. *C. C. MacDuffee* (Madison, Wis.).

Todd, J. A. Combinants of a pencil of quadric surfaces. I. Proc. Cambridge Philos. Soc. 43, 475–487 (1947).

Let $S_\lambda=\lambda_1 S+\lambda_2 S'$ denote the pencil of quaternary quadratic forms defined by the two quadratics $S=x'ax$ and $S'=x'bx$ of matrices a and b, respectively. A covariant of S and S' is a combinant of the pencil S_λ if it is unaltered (except by multiplication by a factor independent of the coefficients) when S and S' are replaced by any two independent linear combinations of S and S'. Such a concomitant is therefore a covariant of the form S_λ regarded as a double form in the two sets of variables x and λ. A complete system of this double form will in general include forms containing λ_1 and λ_2 explicitly. Such forms are called generalized combinants. Let $\{A\}$ be a complete system of a specified type of concomitants of S and S', i.e., of invariants, covariants, contravariants, etc. The author shows that there is a set $\{B_1\}$ of generalized combinants of S_λ the totality of whose coefficients forms a complete system $\{A_1\}$ equivalent to $\{A\}$. (The proof is general and applies to any pencil of forms not necessarily quadratic.) To determine a complete system of combinants of the same type as $\{A\}$ it is only necessary to calculate a complete system of invariants of the forms in $\{B_1\}$.

A complete system of covariants of the two quadratics S and S' consists of the two quadratics S and S', two other quadratic forms d_1 and d_2, the Jacobian G of these four quadratic forms and the five invariants which are the coefficients in the generalized combinant $|\lambda_1 a+\lambda_2 b|$. The system $\{B_1\}$ for covariants consists of (i) G, (ii) $|\lambda_1 a+\lambda_2 b|$, (iii) $\lambda_1 S+\lambda_2 S'$ and (iv) $\lambda_1 Q_1+\lambda_2 Q_2$, where Q_i is determined

as a rational integral function, linear in d_i, of d_i, S, S' and the five invariants. Since G does not involve λ, the set $\{B_1\}$ consists essentially of the quartic (ii) and the two linear forms (iii) and (iv). The complete system of invariants of a binary quartic and two linear forms consists of 20 forms which with G gives a total of 21 covariant combinants of S and S'. The author shows that only one of these 21 forms is reducible (is expressible rationally and integrally in terms of the others) and gives a list of the 20 forms which are irreducible and compose a complete system of covariant combinants of S_λ. The two invariant combinants are of course included in this list. *J. Williamson.*

Todd, J. A. Combinants of a pencil of quadric surfaces. II. Proc. Cambridge Philos. Soc. **43**, 488–490 (1947).

This is a direct continuation of the paper reviewed above, in which contravariant combinants (forms in the tangential coordinates u) are considered. The system $\{B_1\}$ in this case consists of the combinant K, which is the Jacobian of the four quadratic covariants, the quartic $|\lambda_1 a+\lambda_2 b|$ and the cubic which is the tangential equation of the general quadric of the pencil. To the 20 irreducible invariants of a binary cubic and quartic must be added the combinant K. As in the case of the covariants one of the combinants is reducible and two are the two invariants previously determined. The remaining 18 irreducible contravariant combinants which, with the two invariant combinants, form a complete system are listed. *J. Williamson* (Flushing, N. Y.).

Harish-Chandra. On the algebra of the meson matrices. Proc. Cambridge Philos. Soc. **43**, 414–421 (1947).

This paper continues the study of the formalism of the β-matrices introduced by Duffin [Physical Rev. (2) **54**, 1114 (1938)] and Kemmer [Proc. Roy. Soc. London. Ser. A. **173**, 91–116 (1939)] for the description of the meson. In an earlier paper [Proc. Roy. Soc. London. Ser. A. **186**, 502–525 (1946); these Rev. **8**, 302] by the author four 1-row matrices Γ_i^* and four 1-column matrices were introduced. These are now used in the definition of ten 1-row matrices $\Gamma_{\alpha\beta}^*=-\Gamma_{\beta\alpha}^*$ and a similar set of 1-column matrices. Products of these symbols give a basis for the algebra of the 10-row irreducible representation of the β's. The irreducible 5-row representation is also treated. In the second section a similar formalism is established for the Dirac matrices. *W. Givens.*

Littlewood, D. E. An equation of quantum mechanics. Proc. Cambridge Philos. Soc. **43**, 406–413 (1947).

"The complete set of matrix representations of a set of n quantities $\beta_1, \beta_2, \cdots, \beta_n$ satisfying $\beta_i\beta_j\beta_k+\beta_k\beta_j\beta_i=g_{ij}\beta_k+g_{jk}\beta_i$ is obtained for all values of n. It is found that if $n=2\nu$ or $n=2\nu+1$, there are $\nu+1$ irreducible representations with the respective degrees 1, $\binom{n+1}{1}$, $\binom{n+1}{2}$, $\cdots$, $\binom{n+1}{\nu}$, but if $n=2\nu+1$ there are in addition two conjugate representations of order $\binom{n}{\nu}$, the symbol $\binom{n}{\nu}$ denoting the binomial coefficient. The explicit representations are given for $n=2$, 3, 4, 5." [Author's summary.] The method used is to analyze the algebra generated by the β's into its simple matrix subalgebras by expressing the modulus as a sum of irreducible idempotents. In doing this considerable computation is involved. Essentially identical results have been obtained by N. Svartholm [Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 12, no. 9, 94–108 (1942); these Rev. **7**, 4]. *W. Givens.*

Hua, Loo-Keng. Geometries of matrices. II. Study of involutions in the geometry of symmetric matrices. Trans. Amer. Math. Soc. **61**, 193–228 (1947).

The first papers in this series [same Trans. **57**, 441–481, 482–490 (1945), referred to as I and I_1, respectively; these Rev. **7**, 58] laid the foundations for a geometry of symmetric matrices in which the motions of the space are of the form $W=(AZ+B)(CZ+D)^{-1}$ in nonhomogeneous coordinates, where Z and W are symmetric matrices and

$$\mathfrak{T}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a $2n$ by $2n$ symplectic matrix. Regarding two symplectic matrices which differ in sign as a single symplectic transformation, the author proves as his final theorem in this paper that "a topological automorphism of the group formed by the symplectic transformations is either an inner automorphism or an antisymplectic transformation."

Explicit normal forms are found for the involutions and antiinvolutions leaving invariant the fundamental linear complex (null polarity). Thus involutions are found to be of two types according as their product with the null polarity is a second null polarity or is a polarity in a quadric. In the first case a further classification by a signature is required while in the second case any two involutions are equivalent. (The complex number field, used throughout, is here essential.) Canonical forms are $W=HZH$, with H a diagonal matrix with p plus ones and q minus ones ($p=q$) and $W=-Z^{-1}$, respectively. Similarly, antiinvolutions are classified into those of the first and second kinds, satisfying the conditions $\mathfrak{T}\mathfrak{T}^*=\mathfrak{I}$ or $-\mathfrak{I}$, respectively, and a signature is required for those of the second kind. Canonical forms for $Z\rightarrow W$ are $W^*=Z^{-1}$ and $W^*=-HZ^{-1}H$ with H as before and the star denoting the complex conjugate.

The generation of the symplectic group by antiinvolutions is studied and it is shown that every symplectic transformation may be written as a product of either two involutions of the second kind or of four antiinvolutions of the first kind. The conjecture on page 201 that the antiinvolutions of the second kind do not generate the group of symplectic and antisymplectic transformations is false since it would imply the existence of a self-conjugate subgroup of the (simple) symplectic group. In fact, an involution of the second kind can be written as the product of two antiinvolutions of the second kind of signatures zero ($p=q$); for any other signatures more than two can be shown to be required.

Introducing an antiinvolution of the first or second kind as an invariant, the author defines hyperbolic and elliptic subgeometries of the geometry of symmetric matrices. Similarly, if an involution of either sort is required to be invariant, a geometry generalizing a classical type is obtained [cf. p. 210]. After a detailed study of the invariant points of involutions and antiinvolutions, commutative involutions are considered. The main result here may be stated in matrix language as follows. Let $n=2^\sigma\tau$, where τ is odd. The maximum number of n-rowed symplectic, symmetric or skew-symmetric matrices T_i such that $(T_i)^2=-I$ and $T_iT_j=-T_jT_i$ $(i\neq j)$ is $\sigma+2$ $(\sigma\geqq 1)$, $\sigma+1$ $(\sigma\geqq 0)$ or σ $(\sigma\geqq 1)$, respectively, and these maxima are attained. *W. Givens* (Knoxville, Tenn.).

Hua, Loo-Keng. Geometries of matrices. III. Fundamental theorems in the geometries of symmetric matrices. Trans. Amer. Math. Soc. **61**, 229–255 (1947).

In the paper I_1 of this series [cf. the references in the preceding review] the conditions used in I to prove the

generalization of von Staudt's theorem were shown to be redundant for the case of symmetric matrices of order two. The author now proves that the condition of invariance of the harmonic relation may be dispensed with for all orders. The proof involves mathematical induction on the results of I_1 and thus uses chiefly properties of involutions on the projective line rather than results obtained in II.

In the second part of the paper the projective space of symmetric matrices is regarded as an extended space of several complex variables as defined by Osgood [Lehrbuch der Funktionentheorie, v. 2, part 1, 2d ed., Teubner, Leipzig, 1929] and it is proved that "an analytic mapping carrying the extended space onto itself is a symplectic mapping." For the elliptic space of symmetric matrices [cf. the preceding review] the author proves the corresponding theorem: "an analytic automorph of the elliptic space of signature (p, q) is a motion of the space." The theorem for hyperbolic space, due to C. L. Siegel [Amer. J. Math. **65**, 1–86 (1943); these Rev. **4**, 242], is also proved in a new way. *W. Givens* (Knoxville, Tenn.).

Hebroni, Pessach. A process leading to a ring of complex numbers called "continuized matrices." Riveon Lematematika 1, 86–90 (1947). (Hebrew)

Let R be the set of all pairs (a, m) where $a=a(s, t)$ is a function on the unit square and m is a rational number. The set R becomes a ring if addition and multiplication are defined by $(a, m)+(b, n)=(a+b, m+n)$, $(a, m)\cdot(b, n)=(c, mn)$, where $c(s, t)=na(s, t)+mb(s, t)+\int_0^1 a(s, r)b(r, t)dr$. Properties of R and of some related rings are discussed. Applications to integral equations will be given elsewhere.

I. S. Cohen (Philadelphia, Pa.).

Abstract Algebra

Koutský, Karel. Sur les lattices topologiques. C. R. Acad. Sci. Paris **225**, 659–661 (1947).

A topological lattice S is here one with an operation $x\to x^-$ which one may require to be (M) monotone, (A) additive, (U) idempotent, and/or (I) incidental, i.e., $x\leqq x^-$. A $y\epsilon S$ is called an anathema to x if $x\cap y^-=0$. The author now relates the axioms M, A, U, I to six properties of the class of anathemata of each element of S. These relations are especially intimate in the case of complemented modular lattices. *R. Arens* (Los Angeles, Calif.).

Petrescu, Iulian. Topological spaces regarded as lattices with a given endomorphism. C. R. Acad. Sci. Roum. **7**, 11–15 (1945).

The author introduces a closure operation into a lattice by defining it as a homomorphism into a suitable sublattice. Typographical errors and language difficulties obscure many points of the exposition. *R. Arens* (Los Angeles, Calif.).

Loonstra, F. Interprétation topologique d'un théorème de M. Mahler concernant les pseudo-évaluations. Nederl. Akad. Wetensch., Proc. **50**, 862–867 = Indagationes Math. **9**, 373–378 (1947).

If w is the sum of independent pseudo-valuations $w_1, \cdots, w_n$ of a commutative ring R, it was proved by Mahler [Acta Math. **66**, 79–119 (1936)] that the completion R_w of R with respect to w is isomorphic to the direct sum of $R_{w_1}, \cdots, R_{w_n}$. The author proves that the isomorphism is bicontinuous and applies this fact to Hensel's g-adic numbers.

I. S. Cohen (Philadelphia, Pa.).

Kaplansky, Irving. Semi-automorphisms of rings. Duke Math. J. **14**, 521–525 (1947).

Let A and A' be rings with elements a and a', etc. A semi-isomorphism between A and A' is defined by G. Ancochea [J. Reine Angew. Math. **184**, 193–198 (1942); these Rev. **5**, 72] as an isomorphism $a\leftrightarrow a'$ between their additive groups which also satisfies the condition (1) $(ab)'+(ba)'=a'b'+b'a'$. Ancochea has proved recently [Ann. of Math. (2) **48**, 147–153 (1947); these Rev. **8**, 310] that in a simple algebra of characteristic not 2 every semi-automorphism is either an automorphism or an anti-automorphism. In the present paper the author considers rings with unit element and replaces (1) by the condition (2) $(aba)'=a'b'a'$, $(1)'=1'$ which he shows to be equivalent to condition (1) if the characteristic is not 2 and stronger otherwise. He proves that if A and A' are semi-simple rings [in the sense of N. Jacobson [Amer. J. Math. **67**, 300–320 (1945); these Rev. **7**, 2] a semi-isomorphism between them, with condition (2) instead of (1), induces an isomorphism between their centres and then extends Ancochea's result [loc. cit.] to simple algebras of any characteristic. *K. A. Hirsch.*

Kaplansky, Irving. Topological methods in valuation theory. Duke Math. J. **14**, 527–541 (1947).

S'appuyant sur un lemme d'Artin [publié ici pour la première fois] l'auteur donne d'abord une condition nécessaire et suffisante pour que la topologie d'un corps topologique (commutatif ou non) soit définie par une valeur absolue; pour les corps commutatifs, il retrouve ainsi, avec une démonstration plus simple, la condition donnée par I. Shafarevitch [C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 133–135 (1943); ces Rev. **6**, 164]. Cette condition lui donne une nouvelle démonstration du théorème de Pontrjagin-van Dantzig-Jacobson d'après lequel la topologie d'un corps localement compact est définie par une valeur absolue. L'auteur étudie ensuite les corps topologiques métrisables où, pour tout ensemble A auquel O n'est pas adhérent, A^{-1} est borné (un ensemble B étant dit borné si pour tout voisinage U de O, il existe un voisinage V de O tel que $VB\subset U$ et $BV\subset U$); ces corps sont dits "du type V." Son résultat principal est que si A est une algèbre métrique complète sur un corps (commutatif) F de type V, complet et non discret, telle que tout élément de A soit de degré fini sur F, alors les degrés de ces éléments sont bornés (par un nombre indépendant de l'élément); la démonstration est simple (par application du théorème de Baire, suivant une idée de R. Arens) lorsque A est commutative, mais l'auteur ne parvient à traiter le cas général que par l'utilisation de toute la théorie de Jacobson sur les algèbres sans condition minimale. L'article se termine par l'extension aux corps commutatifs de type V de résultats connus sur la complétion des corps valués algébriquement fermés ou réellement fermés. [Le rapporteur signale que la bibliographie est incomplète; en particulier, les corps "de type V" ont déjà été considérés par N. Bourbaki [Eléments de Mathématique, Part I, Livre III, Chap. III–IV, Actual. Sci. Indus., no. 916, Hermann, Paris, 1942, p. 57, ex. 13; ces Rev. **5**, 102] qui démontre entre autres que le complété d'un tel corps est encore un corps.] *J. Dieudonné* (Nancy).

Smiley, Malcolm F. Alternative regular rings without nilpotent elements. Bull. Amer. Math. Soc. **53**, 775–778 (1947).

Un anneau alternatif R est un anneau non associatif tel que $a(ab)=(a^2)b$ identiquement; le sous-anneau de R engendré par deux éléments quelconques est alors associatif.

L'auteur considère les anneaux alternatifs R qui sont réguliers au sens de J. von Neumann, c'est-à-dire tels que pour tout $a\epsilon R$ il existe $x\epsilon R$ tel que $axa=a$; il généralise à ces anneaux un théorème de Forsythe et McCoy [Bull. Amer. Math. Soc. **52**, 523–526 (1946); ces Rev. **7**, 509] d'après lequel une condition nécessaire et suffisante pour qu'un anneau régulier R soit sous-anneau d'un produit d'"anneaux à division" (c'est-à-dire où $ax=b$ et $xa=b$ ont toujours des solutions), est que R n'ait pas d'élément nilpotent. Il en déduit qu'un anneau alternatif R tel que pour tout $a\epsilon R$ il existe un entier $n(a)>1$ tel que $a^{n(a)}=a$, est associatif, et par suite commutatif d'après un résultat de Jacobson [Ann. of Math. (2) **46**, 695–707 (1945); ces Rev. **7**, 238]. En appendice est démontré un théorème de R. H. Bruck, d'après lequel un anneau à division alternatif et commutatif est associatif, et par suite un corps. *J. Dieudonné* (Nancy).

Levitzki, Jakob. On powers with transfinite exponents. II. Riveon Lematematika **2**, 1–7 (1947). (Hebrew)

[Part I appeared in the same Riveon **1**, 8–13 (1946); these Rev. **8**, 193; the terminology and notation of the review of part I are used here.] The transfinite powers A^ρ defined in part I were right powers. That is, A^ρ was defined inductively as follows. (1) If ρ is not a limit number, let $A^\rho=A^{\rho-1}A$; (2) if ρ is a limit number, let $A^\rho=\bigcap_{\tau<\rho}A^\tau$; denote these by $A_r{}^\rho$. In similar fashion, left powers $A_l{}^\rho$ can be defined by replacing (1) by $A_l{}^\rho=AA_l{}^{\rho-1}$. A simple example shows that $A_r{}^\rho$ and $A_l{}^\rho$ need not be the same. Hence in general the left and right kernels of A are different; however, the left and right ultimate kernels are proved equal. The rules $A_r{}^\rho A_r{}^\sigma=A_r{}^{\rho+\sigma}$, etc. do not hold in general, and the equivalences between them are discussed. There are various other results, some counterexamples, and a list of unsolved problems. *I. S. Cohen* (Philadelphia, Pa.).

***Schwarz, Ludwig. Zur Theorie der nichtkommutativen rationalen Funktionen.** Ber. Math.-Tagung Tübingen 1946, pp. 134–136 (1947).

Let K be a ring with unity element. The ring of polynomials of noncommutative indeterminates x_a over K is defined, where for each x_a a subfield L_a of K has been assigned as the set of elements commuting with x_a. In the case that K is a field, a uniqueness theorem is announced for the representation of a noncommutative polynomial as product of a finite number of prime factors. Some results concerning the construction of rational functions are indicated; however, this part of the work has not been completed. *R. Brauer* (Toronto, Ont.).

Snapper, Ernst. Polynomial matrices in several variables. Amer. J. Math. **69**, 622–652 (1947).

Die in einer früheren Arbeit [Amer. J. Math. **69**, 299–326 (1947); diese Rev. **9**, 3] enthaltene Theorie der Moduln mit endlich vielen Erzeugenden über Integritätsbereichen als Koeffizientenbereich wird dazu verwendet, um die Exponentiallösungen von Systemen linearer partieller Differentialgleichungen, deren Koeffizienten Polynome der Unbekannten über einem Grundkörper der Charakteristik 0 sind, vermöge zugeordneter modultheoretischer Invarianten zu beherrschen. Die Theorie der Hilbertschen charakteristischen Funktion [siehe B. L. van der Waerden, Moderne Algebra, Teil 2, Springer, Berlin, 1940] wird auf Moduln mit endlich vielen Erzeugenden über Polynombereichen in endlich vielen Veränderlichen übertragen. Die in der obigen Arbeit von Snapper enthaltenen Kriterien über die Lösbarkeit eines Systemes linearer Gleichungen werden explizit formuliert. *H. Zassenhaus* (Hamburg).

Deuring, Max. Die Anzahl der Typen von Maximalordnungen in einer Quaternionenalgebra von primer Grundzahl. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1945**, 48–50 (1945).

The author derives the formulas for the numbers of ideal classes and types of maximal orders in the rational quaternion algebras $Q_{\infty,p}$ which are ramified at ∞ and at a rational prime p. Furthermore, the formulas for the genus of the extended modular group $\{\text{all } (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \text{ with } c\equiv 0 \pmod p;\ (\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix})\}$ are set in relation with the class numbers of the algebras $Q_{\infty,p}$. The author employs with Hecke ϑ-series of the group $\{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}),\ c\equiv 0 \pmod p)\}$, defined by norm forms of ideals in $Q_{\infty,p}$, and the related integrals of the first kind. The essential hypothesis of the argument is the linear independence of the ϑ-series in question. *O. F. G. Schilling* (Chicago, Ill.).

Deuring, Max. Ein Bemerkung über die Bürmann-Lagrangesche Reihe. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1946**, 33–35 (1946).

Let $f(z)=\sum_{\nu=0}^{\infty}a_{N+\nu}z^{N+\nu}$ and $u=\sum_{i=0}^{\infty}c_iz^i=F(z)$, $c_1\neq 0$, with coefficients $a_{N+\nu}$, c_i in a field k. Then a slight modification of the Bürmann-Lagrange inversion formula asserts that $f(z)=\sum_{n=N}^{\infty}A_nu^n$, where A_n is the coefficient of z^n in the expansion of $f(z)F'(z)(z/F(z))^{n+1}$. The author gives a formal algebraic proof for these formulas which is also valid for fields of prime characteristic p. The proof is based on the linearity of the terms A_n and thus only carried out for $f(z)$ of the form u^m, $m=0, \pm 1, \cdots$. A further extension of the formulas is given for $u=\sum_{j=0}^{\infty}c_{M+j}z^{M+j}$, $c_M\neq 0$, $M>0$, with the additional hypothesis that the expansion of $f(z)$ with respect to integral powers of u exists. In the latter case $M\not\equiv 0 \pmod p$ has to be assumed for prime characteristic. *O. F. G. Schilling* (Chicago, Ill.).

Malcev, A. On solvable Lie algebras. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9**, 329–356 (1945). (Russian. English summary)

An element l of a Lie algebra $\mathfrak{L}$ is called semi-regular if its representing matrix L in the adjoint (regular) representation is similar to a diagonal matrix; l is nilpotent if L is nilpotent, and l is splittable if $l=a+t$, where a is semi-regular and t is nilpotent. If every element of $\mathfrak{L}$ is splittable, then $\mathfrak{L}$ is called splittable. This paper deals primarily with splittable solvable algebras over the field of complex numbers. It is shown that such an algebra can be decomposed as $\mathfrak{A}+\mathfrak{T}$, where $\mathfrak{A}$ is a commutative algebra which is semi-regular in the sense that all of the elements are semi-regular in $\mathfrak{L}$ and $\mathfrak{T}$ is the ideal of nilpotent elements of $\mathfrak{L}$. Conversely, this structure is sufficient that $\mathfrak{L}$ be a splittable solvable algebra. Maximal semi-regular commutative subalgebras of a splittable solvable subalgebra are conjugate. A determination of all solvable splittable algebras with preassigned kernel $\mathfrak{T}$ is given. Also it is shown that the Lie algebra of any algebraic linear group is splittable. This includes a result due to Gantmacher [Rec. Math. [Mat. Sbornik] N.S. **5**(47), 101–146 (1939); these Rev. **1**, 163] to the effect that the derivation algebra of any algebra is splittable. Important for applications is the result proved by the author that any Lie algebra over the complex field

can be imbedded in a splittable Lie algebra and if minimality is assumed the extension is unique in the sense of isomorphism. This result is used to give a simple reduction of Ado's theorem on the representability of Lie algebras by matrices to the case of nilpotent algebras treated by G. Birkhoff [Ann. of Math. (2) **38**, 526–532 (1937)]. Another application is a simple reduction of the first fundamental theorem on invariants of linear groups to the case of nilpotent groups. Finally, an application is given to the study of maximal nilpotent subalgebras of arbitrary Lie algebras. A classification into types is given for such subalgebras and conjugacy is proved for those of the same type. This generalizes a result due to Chevalley [Amer. J. Math. **63**, 785–793 (1941); these Rev. **4**, 2]. *N. Jacobson.*

THEORY OF GROUPS

May, W. Calculation of the stereographic pole figure of the cubic lattice for any given direction. I, II. Nederl. Akad. Wetensch., Proc. **50**, 548–553, 626–639 (1947).

Für die Interpretation von Laue-Diagrammen ist die Umrechnung von dreifach rechtwinkligen Koordinaten auf Gitterkoordinaten und ihre stereographische Projektion wichtig. Dies wird beim kubischen Gitter für die fünf wichtigsten Zonen durchgeführt; beigefügte Tabellen ermöglichen die Interpretation von Diagrammen. *J. J. Burckhardt* (Zürich).

Fumi, F. Sugli operatori matriciali di simmetria macroscopica. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 101–109 (1947).

Fumi, F. Assi di simmetria composta e operatori matriciali di rotazione impropria. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 109–114 (1947).

Im Anschluss an die Arbeiten von F. Seitz [Z. Kristallogr., Mineral. Petrogr. Abt. A. **88**, 433–439 (1934); **90**, 289–313 (1935)] werden die eigentlichen und die uneigentlichen makroskopischen Symmetrien der Kristallographie in Matrizen angegeben und die zugehörigen gruppentheoretischen Sätze bewiesen. Anschliessend stellt der Verf. die mit diesen Symmetrien verbundenen Translationen zusammen. *J. J. Burckhardt* (Zürich).

Stone, M. H. Pseudo-norms and partial orderings in Abelian groups. Ann. of Math. (2) **48**, 851–856 (1947).

To each element $e>0$ of a partially ordered Abelian group, one can associate an "extended" pseudo-norm $\mu(x)$, defined as inf (m/n) such that $-me\leqq nx\leqq me$. Some values of $\mu(x)$ are allowed to be infinite, but not all. Conversely, the author shows that every "extended" pseudo-norm ν is "isomorphic" (i.e., in a bounded ratio) to a pseudo-norm so defined from a partial ordering. In fact, chose e with $1<\nu(e)<+\infty$, and let $x>0$ mean that inf $(1/n)\nu(mx-ne)\leqq 1$, for positive integers m and n. The pseudo-norm defined above from μ is a norm if and only if (A) for every x there exist positive integers m and n such that $me\geqq nx$; (B) the partial ordering $<$ possesses an extension $\lessdot$ such that if n is a positive integer and $nx\lessdot ny$, then $x\lessdot y$, and such that if $nx\,\underline{\lessdot}\,y$ for all positive integers n, then $x\,\underline{\lessdot}\,0$. *G. Birkhoff* (Cambridge, Mass.).

Montgomery, Deane. A theorem on locally Euclidean groups. Ann. of Math. (2) **48**, 650–658 (1947).

Theorem: a locally Euclidean, connected, simply connected group of dimension greater than one contains a proper closed subgroup of positive dimension. This theorem is proved by the following ideas. One can suppose the center of G to be of dimension 0 (otherwise the theorem of Pontrjagin about Abelian groups would give even an Abelian subgroup). The adjoint group of G (group of inner automorphisms) decomposes G into orbits (classes of conjugate elements). Let A_i be the set containing all i-dimensional orbits. Each orbit is connected. Therefore the 0-dimensional orbits must be single points and therefore elements of the center. Thus A_0 must be of dimension zero. If there is an orbit of a dimension less than dim G, e.g., the orbit of the point a, then the transformation $g\rightarrow gag^{-1}$ of G into this orbit reduces the dimension, and therefore must have original sets of positive dimension; one of them is the subgroup of elements of G commuting with a. Hence, if one of the A_i $(0<i<n)$ contains an element, the theorem is true. Now we may suppose A_0 to be of dimension 0 and A_i $(0<i<n)$ to be vacuous. Then A_n must be one single orbit, because G is Euclidean, and the boundary of an n-dimensional orbit must consist of orbits of lower dimensions. The last part of the proof (treating the case of a single orbit A_n) is laborious, but the reviewer remarks that it can be considerably shortened by using a theorem of the reviewer, which states that a group cannot have more than two endpoints. *H. Freudenthal* (Amsterdam).

Bochner, Salomon, and Montgomery, Deane. Groups on analytic manifolds. Ann. of Math. (2) **48**, 659–669 (1947).

The group of all complex analytic homeomorphisms of a compact complex manifold is a Lie group [proved by the authors in the same Ann. (2) **47**, 639–653 (1946); these Rev. **8**, 253]. It is even a complex Lie group, i.e., with complex parameters [proved in the present paper]. For noncompact manifolds this theorem is not true. There is no complex connected group acting in a nontrivial manner with complex parameters, e.g., in any bounded complex Euclidean domain. The compact and the noncompact cases are illustrated by the examples of the elliptic and the hyperbolic geometries.

Another theorem. If in a Riemannian manifold with the topological structure of E_n there acts a compact connected group of analytic transformations with at least one n-dimensional orbit, then this group is essentially a group of orthogonal transformations of E_n. *H. Freudenthal* (Amsterdam).

Dubreil-Jacotin, Marie-Louise. Sur l'immersion d'un semi-groupe dans un groupe. C. R. Acad. Sci. Paris **225**, 787–788 (1947).

Let S be a (noncommutative) semigroup, not having an identity, but satisfying the cancellation law on both sides ($ab=ac$ or $ba=ca$ implies $b=c$). Assume also that S satisfies the following two conditions: (A) if two elements a, b of S have a common right multiple (i.e., $ax=by$ with $x, y\in S$), then one is a left divisor of the other ($au=b$ or $bv=a$); (B) each element of S has only a finite number of left divisors. Then every element of S is uniquely representable as the product of indecomposable elements of S (i.e., elements having no left divisors). Hence S can be imbedded in the free group generated by the indecomposable elements of S. Reviewer's interpretation: the stated conditions characterize a free semigroup. *A. H. Clifford.*

NUMBER THEORY

Nyberg, Michael. Approximating fractions for square roots derived by geometrical considerations. Norsk Mat. Tidsskr. **29**, 75–79 (1947). (Norwegian)

Skolem, Th. A remark on the preceding article of M. Nyberg. Norsk Mat. Tidsskr. **29**, 79 (1947). (Norwegian)

Cotlar, Mischa. A method for obtaining congruences of Bernoulli numbers. Math. Notae **7**, 1–29 (1947). (Spanish)

The author has previously defined sets of polynomials $\phi_n(x)$, $g_n(x)$, $M_n(x)$ [Math. Notae **6**, 69–95 (1946); these Rev. **9**, 30]. In particular,

$$\frac{1}{1-t}\left(\frac{1+t}{1-t}\right)^x=\sum_{n=0}^{\infty}M_n(x)t^n,$$

and $M_n(x)=M_n{}^0x^n+\cdots+M_n{}^n$ $(M_n{}^n=1)$. Typical results of the present paper are the following. Let p be an odd prime and $2\leqq 2k\leqq p-1$; then

$$\frac{2(2^{2k}-1)}{2k}B_{2k}\equiv M_{p-1}^{2k-1}\pmod p,$$

which for $2k=p-1$ reduces to a known result. For $2k<p-1$, $B_{2k}/2k\equiv\frac{1}{2}M_p^{2k}\equiv-(2^{2k}-2)^{-1}M_p^{2k+1}/p$, with a like congruence for $2k=p-1$. A more complicated formula is given which involves the quotients of Fermat and Wilson; as a corollary the following is derived:

$$s^{2k-1}\sum_{a=1}^{p-1}(-1)^{r(sa)}a^{2k-1}\equiv-\frac{2(2^{2k}-1)}{2k}B_{2k}\pmod p,$$

where $s\leqq p-1$ and $r(a)=a-p[a/p]$. *L. Carlitz.*

Bell, E. T. The problems of congruent numbers and concordant forms. Proc. Nat. Acad. Sci. U. S. A. **33**, 326–328 (1947).

The author indicates necessary and sufficient forms of r, m, s, n in order that there shall exist X, Y, Z, W all different from zero, satisfying $rX^2+mY^2=rZ^2$, $sX^2+nY^2=sW^2$, where all letters denote rational integers and solutions are complete solutions in such integers. Altogether there are 41 parameters. Each of m, n is of degree 3 in these parameters; each of r, s is of degree 71; each of X, Z, W is of degree 49, and Y is of degree 83. It is noted that for $r=s=Y^2=1$, $n=-m$ (m a given constant) and for $r=s=1$ the Diophantine system above includes a classical problem on congruent numbers and concordant forms, respectively. Both problems (the first for m arbitrarily assigned) are still unsolved. The author remarks that "the inherent complexity of the solution" given in his paper "may suggest why these two old and apparently simple problems are still not completely solved." *W. H. Gage* (Vancouver, B. C.).

Min, S. H. On systems of algebraic equations and certain multiple exponential sums. Quart. J. Math., Oxford Ser. **18**, 133–142 (1947).

Let $f(x_1,\cdots,x_n)$ be a polynomial of degree k with integer coefficients. Consider a field K of p^m elements (p prime), assume f of degree k in K, and construct the sum $S\{f,K\}=\sum\exp\{2\pi i\mathfrak{S}(f)/p\}$, where $\mathfrak{S}(f)=\mathfrak{S}[f(x_1,\cdots,x_n)]$ denotes the trace of $f(x_1,\cdots,x_n)$, and $x_1,\cdots,x_n$ run independently over all elements of K. Write

$$f(\alpha_1t+\beta_1,\cdots,\alpha_nt+\beta_n)=\sum_{r=0}^{k}F_r(\alpha_1,\cdots,\alpha_n;\beta_1,\cdots,\beta_n)t^r,$$

and denote by M the matrix of k rows and $2n$ columns, with the ith row $\partial F_i/\partial\alpha_1,\cdots,\partial F_i/\partial\alpha_n,\partial F_i/\partial\beta_1,\cdots,\partial F_i/\partial\beta_n$. The principal theorem proved is that, if $k\geqq 2n$ and not every minor determinant of order $2n$ in M vanishes identically, then for large p, $S\{f,K\}=O(p^{mn(1-1/k)})$, where the constant implied by O depends only on k and n. The case $m=n=1$ reduces to Mordell's estimate for the exponential sum $\sum\exp 2\pi if(x)/p$ [Quart. J. Math., Oxford Ser. **3**, 161–167 (1932)]. The case $n=2$ was proved by Hua and Min [Acad. Sinica Science Record **1**, 23–25 (1942); these Rev. **5**, 255].

The proof is based in part on a theorem concerning the number of solutions of a system of polynomial equations $f_i(x_1,\cdots,x_n)=0$ $(i=1,\cdots,m)$ in a field F. A solution in which the Jacobian matrix $(\partial f_i/\partial x_j)$ is of rank r is called a solution of rank r; if $r=n$ the solution is termed nonsingular. The theorem is that the number of nonsingular solutions is $O(1)$, where the constant implied by O depends only on n, m, and the degrees of the f_i. It is proved also that the number of solutions of rank r can be arranged in $O(1)$ sets, such that if a certain $n-r$ variables are given the remaining variables have only $O(1)$ possibilities. A similar result is proved for nonproportional solutions in the case of homogeneous equations. *G. Pall* (Chicago, Ill.).

***Hasse, Helmut. Über die Berechnung der Klassenzahl reeller abelscher Zahlkörper.** Ber. Math.-Tagung Tübingen 1946, pp. 74–75 (1947).

The starting point is the formula for the class number h of an algebraic number field K, derived by analytic methods. The field K is assumed to be real and it is further assumed that either K is cyclic over the field R of rational numbers, or that K is Abelian and that every prime divisor of the discriminant is a power of a prime ideal of K. The author announces that he has developed methods in these two cases which allow h to be written in an arithmetical form, similar to that known for the special case where K is the maximal real subfield of the field of the pth roots of unity, p a prime. He also mentions that he has determined the class numbers of all real cyclic fields of degrees 3 and 4 with a conductor $f\leqq 100$. *R. Brauer* (Toronto, Ont.).

Fogels, E. Zur arithmetik quadratischer Zahlenkörper. Univ. Riga. Wiss. Abh. Kl. Math. Abt. **1**, 23–47 (1943). (German. Latvian summary)

The author calls an integer of the quadratic field $R(\sqrt{(-5)})$ normal if it is uniquely factorable, apart from order of factors and units, as a product of indecomposable integers of the field, and anormal otherwise. He proves that almost all rational integers a, and almost all integers $\alpha=a+b\sqrt{(-5)}$ are anormal, in the sense that, if $f(a)$ $[f(\alpha)]=1$ or 0 according as a $[\alpha]$ is normal or anormal, the mean value of $f(a)$ $[f(\alpha)]$ for $a\leqq x$ $[N(\alpha)=a^2+5b^2\leqq x]$ tends to zero as $x\to\infty$. The proofs employ the representation theory for the form x^2+5y^2, on the basis of which Kummer ideal numbers are

introduced, all normal rational integers are determined and exact expressions or simple bounds are found for the numbers of integers α of given norm n, according to the occurrence of primes 2, 5, $p\equiv 1, 9$, $q\equiv 3, 7$, $r\equiv 11, 13, 17, 19 \pmod{20}$, in the factorization of n. Asymptotic approximations for sums of coefficients of certain Dirichlet series are also used. The proofs could be carried through for other imaginary quadratic fields having a single class in each genus. Other complications, due to the infinity of units, would arise in the case of real quadratic fields. *R. Hull.*

Courbatoff, V. Généralisation d'un théorème de Schur sur une classe de fonctions algébriques. Rec. Math. [Mat. Sbornik] N.S. **21**(63), 133–141 (1947). (Russian. French summary)

Un polynome $f(x)$ à coefficients entiers rationnels est dit substitutif pour un nombre premier p si $f(x)\equiv f(y) \pmod p$ implique, pour les x, y entiers rationnels, $x\equiv y \pmod p$. I. Schur a démontré [S.-B. Preuss. Akad. Wiss. **1923**, 123–124] que, si $f(x)$ est substitutif pour une infinité de nombres premiers, parmi les branches $y_1(x), y_2(x), \cdots, y_n(x)$ de la fonction algébrique $y(x)$, définie par l'équation $f(y)=x$, il y a au plus $n-2$ linéairement indépendantes [n est le degré de $f(x)$]. L'auteur démontre que si $f(x)$ a la forme $x^n+\cdots+a_kx^{n-k}+\cdots+a_n$, où $a_k\neq 0$, entre les $y_1(x)^\lambda, y_2(x)^\lambda, \cdots, y_n(x)^\lambda$, quel que soit $\lambda\leqq k$, il y a au moins une relation linéaire homogène à coefficients constants, distincte de $y_1(x)^\lambda+y_2(x)^\lambda+\cdots+y_n(x)^\lambda=-\lambda a_\lambda$ ($=0$, si $\lambda<k$). Pour sa démonstration, l'auteur se sert du fait évident, remarqué par Hermite, que si $f(x)$ est substitutif pour p, la somme des coefficients des $x^{a(p-1)}$ dans $f(x)^m$ est, pour tout $m<p-1$, congrue à 0 (mod p). Posons $\lambda=[mn/(p-1)]$, $p=nt-r$ $(0\leqq r<n)$, $m=\nu+\lambda t$ et notons $A_q^{(\lambda,m)}$ le coefficient de x^q dans $f(x)^m$. Le théorème de Hermite donne $\sum_\lambda A_{\lambda(p-1)}^{(\lambda,m)}\equiv 0 \pmod p$. L'auteur calcule $A_{\lambda(p-1)}^{(\lambda,m)}$ et montre qu'il est congrû (mod p), quand $\lambda\leqq k$, où $f(x)=x^n+\cdots+a_kx^{n-k}+\cdots$, au coefficient $d_{n\nu+\lambda r+\lambda}$ de $z^{n\nu+\lambda r+\lambda}$ dans $[z^nf(1/z)]^{\nu+\lambda r/n}$. D'autre part, il montre que, si $h<\lambda$, il existe un $\lambda'<\lambda$ et un $m'<p-1$ tels que $A_{h(p-1)}^{(\lambda,m)}\equiv A_{\lambda'(p-1)}^{(\lambda',m')} \pmod p$, d'où il résulte, par induction, $A_{\lambda(p-1)}^{(\lambda,m)}\equiv 0 \pmod p$. Il en résulte facilement que, si r est tel que $f(x)$ soit substitutif pour une infinité de $p\equiv -r \pmod n$, ce coefficient rationnel est divisible, pour $\lambda\leqq k$, par tous ces nombres premiers, donc est nul. Or il se trouve que si $y(x)=\sum_{q=-\lambda}^{\infty}c_q^{(\lambda)}x^{q/n}$ est le développement de $y(x)^\lambda=y_1(x)^\lambda$ au voisinage de ∞, on a $c_q^{(\lambda)}=d_{q+\lambda}$. Donc, tous les termes de $y(x)^\lambda$ dont l'exposant est congrû à $\lambda r/n$ (mod 1) sont nuls, et, par suite, si ϵ_i est la racine n-ième de 1 telle que $y_i(x)=\sum_{q=-1}^{\infty}c_q^{(1)}\epsilon_i^q x^{q/n}$, la résolvante de Lagrange $\sum\epsilon_i^{-\lambda r}y_i(x)^\lambda$ est nulle, ce qui est la relation cherchée. *M. Krasner.*

Cohen, Eckford. Sums of an even number of squares in $GF[p^n, x]$. **II.** Duke Math. J. **14**, 543–557 (1947).

The case $m=0$ excluded in the author's first paper with the same title [same J. **14**, 251–267 (1947); these Rev. **9**, 81] is disposed of here. It is proved that, if $F\epsilon GF[p^n, x]$, $\deg F<2k$, for a fixed $k\geqq 1$, and if $\alpha_1, \cdots, \alpha_{2s}$ are nonzero elements of $GF(p^n)$, then the number of solutions of $\alpha_1X_1^2+\cdots+\alpha_{2s}X_{2s}^2=F$, $X_i\epsilon GF[p^n, x]$, $\deg X_i<k$, $i=1, \cdots, 2s$, is $R_{s-1}(F, \zeta)$, where $\zeta=+1$ or -1 according as $(-1)^s\alpha_1\cdots\alpha_{2s}$ is or is not a square in $GF(p^n)$, and where $R_t(M, \mu)$ is defined by

$$R_t(M, \mu)=\{p^{n(t+1)}-\mu\}p^{nt(2k-1)}\sum_{s=0}^{k-1}\mu^s p^{-nst}g_s(M), \qquad M\neq 0,$$

$$R_t(0, \mu)=p^{2kn(t+1)}+\{p^{n(t+1)}-\mu\}\sum_{s=0}^{k-1}\mu^s p^{-nst}g_s(0),$$

$$g_s(M)=\delta_s(M)-\delta_{2k-s-1}(M),$$

where $\delta_s(M)=0$ for $s<0$, and $\delta_s(M)$ is the number of distinct elements of $GF[p^n, x]$, of leading coefficient (sgn) 1, which divide M for $s\geqq 0$. The case $k=1$ is equivalent to a theorem of Jordan [Traité des Substitutions Algébriques, Paris, 1870, nos. 197–200] for $n=1$, generalized by Dickson [Linear Groups, Leipzig, 1901, § 65], on the number of solutions of $\alpha_1\xi_1^2+\cdots+\alpha_{2s}\xi_{2s}^2=\kappa$ in $GF(p^n)$. *R. Hull.*

Carlitz, L., and Cohen, Eckford. Cauchy products of divisor functions in $GF[p^n, x]$. Duke Math. J. **14**, 707–722 (1947).

In this paper, numerical relations for divisor functions which have already been proved by the authors [cf. the preceding review and same J. **14**, 13–20 (1947); these Rev. **8**, 503], and some additional ones, are shown to lead in certain cases to rings whose elements are divisor functions of appropriate sets, with the operations of ordinary addition and Cauchy multiplication. The divisor functions are of the type

$$\varphi(M)=\sum_{i=0}^{r}a_i\delta_i(M), \tag{1}$$

where r ($=2k$ or $2k+1$) is a fixed nonnegative integer and the a's are arbitrary complex numbers. Relative to a fixed $F\epsilon GF[p^n, x]$, $\deg F=f\leqq r$, three Cauchy products are defined. Let α and β be fixed nonzero elements of $GF(p^n)$ such that $\alpha+\beta=\operatorname{sgn} F$ or 0 according as $f=r$ or $f<r$. The C_i-products, $i=1, 2, 3$, for φ, ψ as in (1), are:

$$C_1\colon \varphi\cdot\psi=\sum\nolimits_1\varphi(A)\psi(B)=\xi(F),$$

with $\sum_1$-summation over pairs A, B such that $\deg A=\deg B=r$, $\operatorname{sgn} A=\alpha$, $\operatorname{sgn} B=\beta$, $A+B=F$;

$$C_2\colon \varphi\cdot\psi=\sum\nolimits_2\varphi(A)\psi(B)=\xi(F), \qquad f=r,$$

with $\sum_2$-summation over pairs A, B such that $\deg A=r$, $\deg B<r$, $A+B=F$;

$$C_3\colon \varphi\cdot\psi=\sum\nolimits_3\varphi(A)\psi(B)=\xi(F), \qquad f<r,$$

with $\sum_3$-summation over pairs A, B such that $\deg A<r$, $\deg B<r$, $A+B=F$. The orthogonality relations satisfied by the divisor functions $\gamma_s(M)=\delta_s(M)-\delta_{s-1}(M)$ $(s\geqq 0)$:

$$\sum\nolimits_{1,2,3}\gamma_i(A)\gamma_j(B)=\begin{cases}0, & i\neq j,\\ p^{n(r-i)}\gamma_i(F) & i=j,\end{cases}$$

for the $C_{1,2,3}$-products, where $i\geqq 0$, $j\geqq 0$, $i+j\leqq r$, i and j are fixed, are extensively employed, as is the relation $\delta_i(M)=\delta_{r-i}(M)$ for $\deg M=r$. It is proved, for example, that the linear set $a_0\gamma_0+\cdots+a_k\gamma_k$ constitutes a ring with unity element with respect to ordinary addition and C_1-multiplication for $f=r$. The C_2-product, and the C_1-product for $f<r$, do not yield rings. The C_3-product yields a ring which is the direct sum of $r-1$ complex fields and whose unity element is $R_{-1}(M, 1)$ of the preceding review, written formally as an element of the ring. *R. Hull.*

ANALYSIS

Calculus

De Cicco, John. An extension of Euler's theorem of homogeneous functions. Scripta Math. **13**, 48–52 (1947).

It is proved that a function $\phi(x, y)$ can be expressed as the sum of r functions $f_{n-r+1}(x, y), f_{n-r+2}(x, y), \cdots, f_n(x, y)$, where $f_k(x, y)$ is homogeneous of degree k in (x, y), if and only if ϕ satisfies the differential equation

$$\sum_{k=0}^{r} {}_rC_k(-1)^k{}_{n-k}P_{r-k}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{(k)}\phi=0.$$

As noted by the author, this result and the proof of it can readily be extended to the case of m variables. For $r=1$, the differential equation reduces to the Euler equation for a homogeneous function of degree n. The author indicates how his result may be applied to a problem suggested by Kasner [Amer. J. Math. **26**, 164–168 (1904)] concerning polars of degree r of any point with respect to an algebraic curve of degree n. *W. H. Gage* (Vancouver, B. C.).

Arešev, M. S. On the differentiation of composite functions. Doklady Akad. Nauk SSSR (N.S.) **57**, 311–313 (1947). (Russian)

The author considers the problem of representing the nth iterated total differential d^nf, where $f=f(u_1, \cdots, u_m)$ and $u_k=u_k(x_1, \cdots, x_l)$, in terms of $\partial f/\partial u_k$ and $\partial u_k/\partial x_i$. He introduces the operator

$$L_r=\sum_{i=1}^{m}d^ru_i\frac{\partial}{\partial u_i}$$

and obtains d^nf as a polynomial in L_rf. Formulas (6) and (10) are confusing since the range of summation is not adequately specified. *R. Bellman* (Princeton, N. J.).

Kimball, W. S. True partial derivatives of derivatives. Philos. Mag. (7) **38**, 32–45 (1947).

In two previous papers [same Mag. (7) **30**, 190–222 (1940); **32**, 137–154 (1941); these Rev. **2**, 78; **3**, 145] the author developed formulas for the partial derivatives with respect to x and y of the derivatives of a function $y=\varphi(x)$. The present article gives some extensions of these formulas, including an expansion in terms of their partial derivatives and an expression for finite variations of these derivatives. The work is formal in character; see the reviews of the earlier papers, cited above, for general comment.
A. Dresden (Swarthmore, Pa.).

Mahajani, G. S., and Behari, Ram. An interesting result in the logarithmic expansion. J. Univ. Bombay (N.S.) **15**, part 5, Sect. A., 1–2 (1947).

In this paper the authors generalize Maclaurin's expansion of $\log(1+x)$, replacing x by a power series $a_1x+a_2x^2+\cdots$. Proof is deferred to a later paper. [Misprints were noted in the formula for D_2; on the second page Δ_3 is written where Δ_4 is intended.] *T. Fort* (Athens, Ga.).

Theory of Sets, Theory of Functions of Real Variables

Dushnik, Ben. Maximal sums of ordinals. Trans. Amer. Math. Soc. **62**, 240–247 (1947).

Let a_α, $\alpha<\beta$, be an ordinal sequence of ordinals, not necessarily arranged according to order, and with repetitions allowed. Every one-to-one map $\alpha(\alpha')$ of the ordinals less than β onto themselves reindexes the system a_α as $a_{\alpha(\alpha')}$, and defines a sum $\sum_{\alpha'<\beta}a_{\alpha(\alpha')}$. The author shows that the class of sums thus obtained always contains a maximum if β is any limiting ordinal of Cantor's second number class, but that if β is a nonlimiting ordinal of Cantor's second number class, or is any larger ordinal, then there exists a sequence of length β for which the corresponding class of sums contains no maximum. *L. H. Loomis.*

Vázquez García, R., and Zubierta Russi, F. The cardinal number of complete linear homogeneous continua. Bol. Soc. Mat. Mexicana **2**, 91–93 (1945). (Spanish)

Pursuing the investigation of a notion introduced by Birkhoff [Vázquez and Zubierta, same Bol. **1**, no. 2, 1–14 (1944); these Rev. **5**, 231] the authors prove that a complete linear homogeneous continuum has the power of the real number system. *R. Arens* (Los Angeles, Calif.).

Sierpiński, Wacław. Sur les images de classe 1 d'ensembles linéaires. Fund. Math. **34**, 163–165 (1947).

Let $f(x)$ be a real function of Baire class α on a linear set X. The set $f(X)$ is called an "image of class α" of X. The author proves that every image of class 2 of X is an image of class 1 of an image of class 1 of X. This theorem can be generalized to images of class n (finite), which can be represented as n-fold superposed images of class 1.
A. Rosenthal (Lafayette, Ind.).

Kondô, Motokiti. Sur les opérations analytiques dans la théorie des ensembles et quelques problèmes qui s'y rattachent. II. J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **10**, 35–76 (1941).

[The first part, devoted to a study of analytic operations of sets, was published in the same J. **7**, 1–34 (1938).] In this second part the author introduces the notion of analytic operations of functions. Again the author's discussions are closely related to the work of L. Kantorovitch and E. Livenson [Fund. Math. **18**, 214–279 (1932); **20**, 54–97 (1933)]. Chapter VIII studies operations of real numbers. By means of the bounding transformation the author defines the continuity of an operation on a sequence of numbers. Moreover, he calls a continuous operation $\Phi(x_n)$ "topologic" if, for every continuous monotone increasing function $\chi(t)$ in $[-\infty, +\infty]$ we have $\Phi(\chi(x_n))=\chi(\Phi(x_n))$. If a set $\mathfrak{N}$ of irrational numbers $(n_1, n_2, \cdots)$ is given, he calls the operation $\Phi(x_n)=\sup_{\mathfrak{N}}\{\inf_k(x_{n_k})\}$ a "Hausdorff" operation and $\mathfrak{N}$ its base. The author then proves the theorem: in order that an operation $\Phi(x_n)$ of numbers be topologic, it is necessary and sufficient that it be a Hausdorff operation. Moreover, the bases of topologic operations and the representation of the operations of numbers are discussed.

Chapter IX introduces, and then studies, the analytic operations of functions $\Phi(F_n(x))$ using operations $\Phi(x_n)$ of numbers. The author discusses the relation between analytic operations of functions and of sets. The main aim of this chapter is the representation of the analytic operations of functions by means of closed sieves of functions and, more generally, by means of functional sieves of functions. Generalizations to the so-called quasi-analytic operations of functions are indicated. Chapter X discusses general prop-

erties of the analytic operations of functions and of closed sieves of functions. In particular, theorems of existence and of extension are proved. *A. Rosenthal* (Lafayette, Ind.).

Hadwiger, H. Bemerkung zu einer Grössenrelation bei Punktmengen. Portugaliae Math. 6, 45–48 (1947).

Let $A<B$ (A and B being sets in a metric space) mean that there is a one-to-one mapping φ, such that $\varphi(A)=B$, $\rho[\varphi(x), \varphi(y)]>\rho(x, y)$ for all x, y of A. It is shown that $A<A$ is false if A is a bounded set in R_n (the proof applies to a "totally bounded" set in any metric space). An easy example is given of noncompact sets A, B with $A<B$, diam $(A)=$ diam (B). *A. J. Ward* (Cambridge, England).

Pauc, Chr. Topologie des contingents et paratingents. Rend. Circ. Mat. Palermo 62, 137–238 (1940).

For any set E define the reduced nth power $E^n_{r.}$ as the set of all different ordered n-tuples $(e_1, \cdots, e_n)$, $e_i \epsilon E$, and the reduced symmetric nth power $E^n_{r.s.}$ as the set of all different nonordered n-tuples $(e_1, \cdots, e_n)$. Let K denote generally a continuum (compact connected set with more than one point) in a metric space R. Then $K^n_{r.s.}$ is connected; $K^n_{r.}$ is connected unless K is an arc or a topological circle in which cases $K^n_{r.}$ has $n!$ and $(n-1)!$ components, respectively. In the long and involved proof the smallest cardinal $\nu(K)$ is used, for which K ceases to be connected after removal of ν arbitrary different points. If $\nu(K)$ is finite then K is an ordinary curve [gewöhnliche Kurve in the terminology of Menger, Kurventheorie, Teubner, Leipzig-Berlin, 1932 (quoted as M), p. 63] and $\nu(K)$ equals the number of extremities plus the Euler characteristic of K. If $\nu(K)\leqq\aleph_0$ then K has no accumulation continuum [Häufungskontinuum, M. p. 264].

Let $\theta(y)$ be a continuous mapping of $R^n_{r.}$ in an abstract limit space Γ (if R is E^2 then the line $\theta(y)$ through a pair $y=(x_1, x_2)$ of R is a function $\theta(y)$ on $R^2_{r.}$ after the set of all lines in R has been topologized). We define $\theta^{\sup}(y)=\sum \limsup \theta(y_\nu)$, where the summation is extended over all sequences $y_\nu \to y$, and y is in R^n but not necessarily in $R^n_{r.}$. For $y \epsilon R^n_r$ the continuity of θ implies $\theta^{\sup}(y)=\theta(y)$ (in the above example tangents of a set are obtained from the secants by the introduction of $\theta^{\sup}$). If y_ν is restricted to a subset M of R the correspondingly defined function is denoted by $\theta^{\sup}_M(y)$. For symmetric θ the set $\theta^{\sup}_K(y)$ is connected. If $y=(x_1, \cdots, x_n)$ is an n-tuple of points consisting of the points $x^1, \cdots, x^s$ with multiplicities $n_1, \cdots, n_s$, and if x^{i_j}, $j=1, \cdots, t$, are the x^k at which a neighborhood of K is an arc, then the number of components of $\theta^{\sup}_K(y)$ is at most $n_{i_1}! \cdots n_{i_t}!$.

If $x_1=\cdots=x_n=x$ then $\theta^{\sup}_M(y)=P^n_M(x)$ is called the θ-paratingent of M at x. If θ is symmetric then $P^n_M(x)$ is connected. In the general case, disconnectedness of $P^n_M(x)$ implies that K is an arc in the neighborhood of x. The θ-contingent $C^n_M(x)$ of M at x is defined in an analogous manner; $C^2_k(x)$ is connected for every part x which is not a local cut point [lokaler Zerlegungspunkt, M. p. 164]. If $C^2_k(x)$ is disconnected at every point of K then K is a regular curve [reguläre Kurve, M. p. 96]. If both R and Γ are compact and metric then the correspondence $x \to C_M(x)$ of R in 2^Γ is (for fixed M) of the fourth Lusin class.

H. Busemann (Los Angeles, Calif.).

Arešev, M. S. On certain local properties of functions of two independent variables. Doklady Akad. Nauk SSSR (N.S.) 57, 423–426 (1947). (Russian)

For the study of a function $f(x, y)$ in the vicinity of point (a, b) (all numbers are real) introduce the notations:

$$\varphi=a+\sum_{k=1}^{r}(a_k/k!)u^k, \quad y_m(v)=b+\sum_{k=1}^{m-1}(b_k/k!)u^k+(u^m/m!)v.$$

The curve $x(u)$, $y(u)$ is said to belong to the neighborhood $N_m=(\varphi, b, b_1, \cdots, b_m)$ of (a, b) if $x(u)=\varphi(u)$ and $y(u)$ has the form $y_m(\lambda(u))$ where $\lambda(u)\to b_m$ for $u\to 0$. Various theorems on the solutions of $f(x, y)=0$ in the neighborhood of (a, b) are stated without proof, the strongest of which is the following. Let $f(x, y)$ satisfy the five conditions: (1) $f(x, y)$ has continuous partial derivatives of order $l+q+1$; (2) the ith derivative of $f(x(u), y(u))$ with respect to u does not contain a term in y^{m+1} for $i=1, \cdots, l$; $d^ky/du^k=b_k$, $k=1, \cdots, m$; (3) $(d^i/du^i)f(\varphi, y_m(v))_{u=0}=0$ for $i=0, \cdots, l-1$; (4) $v=b_m$ is a q-fold root of the polynomial $(d^l/du^l)f(\varphi, y_m(v))_{u=0}$; (5) $(d^{l+i}/du^{l+i})f(\varphi, y_{m+1}(v))_{u=0}=0$ for $i=0, \cdots, q-1$. Then each solution $x(u)$, $y(u)$ of $f=0$ which belongs to N_m can be composed of curves which belong to N_{m+1}, where b_{m+1} is a real root of $(d^{l+q}/du^{l+q})f(\varphi, y_{m+1}(v))_{u=0}=0$.

H. Busemann (Los Angeles, Calif.).

Kappos, Demetrios A. Die Cartesischen Produkte und die Multiplikation von Massfunktionen in Booleschen Algebren. I. Math. Ann. 120, 43–74 (1947).

Imitating a well-known method of defining measurability and measure in product spaces (proceeding from rectangles through disjoint finite unions of rectangles to the metric completion of the class of such unions), the author defines Cartesian product (with a finite number of factors) for Boolean algebras and measure algebras. *P. R. Halmos.*

Barricelli, Nils Aall. L'intégrale relative d'une fonctionnelle et ses applications dans la théorie de la distribution de probabilité d'une courbe. Arch. Math. Naturvid. 49, no. 3, 35–117 (1947).

The author discusses certain integrals and mean values in the space of real functions on a closed interval $[a, b]$. If F is a functional (i.e., a function of the elements $x(t)$ in the function space) and F_n is a function of n real variables, if $a<t_1<\cdots<t_n<b$, and if $\lim_{n\to\infty} F_n(x(t_1), \cdots, x(t_n))=F(x)$, then the sequence $\{F_n\}$ is said to be asymptotic to F. The "integrals" and "relative integrals" studied are defined as limits of finite dimensional integrals of the terms of asymptotic sequences. After deriving some of the elementary properties of such integrals (additivity, transformation formulas, etc.), the author turns to the problem of evaluating some of them, and to applying his results to the problem of the gambler's ruin, the calculation of the probability of the propagation of mutations, the molecular theory of gases, and meteorology.

The formulations of the author's definitions and results are not always rigorous. The reader is warned that the functions considered need not be continuous, but their exact domain is never specified; at times it seems to be assumed that only a finite number of discontinuities can occur, while at other times the functions in question must be differentiable. The validity of the limit process which is offered to circumvent the obstacle of nondifferentiability is neither obvious to the reviewer nor proved. The author remarks of a certain quantity that it is "un produit d'un nombre infini

de grandeurs généralement différents de 1, et il est par conséquent infini ou nul." The notation is cumbersome and the major part of the paper consists of lengthy analytic and numerical computations. The author's applications appear to be amenable to treatment by the more conventional methods of the theory of probability and stochastic processes. Except for a mention of an earlier paper of the author [same Arch. **45**, no. 12, 131–154 (1942); these Rev. **7**, 123], the paper contains no references to the literature. There are numerous misprints. *P. R. Halmos.*

Ionescu Tulcea, C. T. Sur l'intégration des nombres dérivés. C. R. Acad. Sci. Paris **225**, 558–560 (1947).

A definition of the Denjoy (Perron) integral based on right and left major functions and right and left minor functions has been given by McShane in his book [Integration, Princeton University Press, 1944; these Rev. **6**, 43]. The author shows that if only right major and left minor (left major and right minor) functions are used there is obtained a definition which gives not only the Denjoy and Perron integrals but also other integrals, D^*-integrals (D_*-integrals). The function $F(x)$ is AC_+ (AC^+, AC_-, AC^-) on a set E if for $\epsilon>0$ there is a $\delta>0$ such that if (a_k, b_k) is any set of nonoverlapping intervals with a_k, b_k points of E with $\sum(b_k-a_k)<\delta$, then $\sum[F(x_k)-F(a_k)]>-\epsilon$ (or $\sum[F(x_k)-F(a_k)]<\epsilon$, $\sum[F(b_k)-F(x_k)]>-\epsilon$, $\sum[F(b_k)-F(x_k)]<\epsilon$), $a_k\leqq x_k\leqq b_k$. If an interval I is covered by a sequence of closed sets on each of which $F(x)$ is AC_+ (AC^+, AC_-, AC^-) and if, almost everywhere on I, $D_+F=D^-F=f$ ($D^+F=D_-F=f$) then F is the D^*-integral (D_*-integral) of f on I. *R. L. Jeffery.*

Tagamlickiĭ, Ya. A. On the integration of sequences of functions. Doklady Akad. Nauk SSSR (N.S.) **57**, 17–19 (1947). (Russian)

Let E be a measurable point-set of finite Lebesgue measure in Euclidean n-space, and let f_n ($n=1, 2, \cdots$) be a sequence of summable functions in E which converge asymptotically to f [i.e., for each $\epsilon>0$ the measure of the set $|f_n-f|>\epsilon$ tends to 0 as $n\rightarrow\infty$]. Denote by $I_n(e)$ the integral of f_n on the measurable subset $e\subset E$. The author establishes two equivalent conditions (I) and (II), each of which is necessary and sufficient in order that f be summable and that $\int f_n\rightarrow\int f$ on every $e\subset E$. (I) For each infinite selection $n=n_k$ of the f_n, there exists a summable $\varphi\geqq 0$ so that $|f_n|\leqq\varphi$ for some arbitrarily large $n=n_k$. (II) Each infinite selection $n=n_k$ of the $I_n(e)$ has at least one absolutely continuous limit-function $J(e)$, where (for each e) $J(e)$ is any limit of the selection $n=n_k$ of the numbers $I_n(e)$. *L. C. Young.*

Theory of Functions of Complex Variables

Serghiesco, Stéphan. Sur le nombre des zéros et des pôles distincts d'une fonction méromorphe dans un contour fermé. C. R. Acad. Sci. Paris **225**, 485–487 (1947).

The integral

$$I=(2\pi i)^{-1}\int_C \frac{F'^2-FF''}{FF'}dz$$

gives the number of distinct poles and zeros of $F=g/h$ inside of C, diminished by the number of zeros of $\Phi=gh'-hg'$ which are not zeros of gh [cf. the author's previous note, same C. R. **224**, 440–442 (1947); these Rev. **8**, 453]. One will naturally try to gain additional information concerning the disturbing term. The author remarks that Φ is a differential invariant of the bundle $\lambda g+\mu h$, which in the case of rational g, h has been studied by Hilbert; it is known that only a limited number of bundles have the same Φ. Remarks are made on the number of combinations $\lambda g+\mu h$ with at least one multiple factor. *L. Ahlfors* (Cambridge, Mass.).

Graetzer, H. Note on power series. J. London Math. Soc. **22**, 90–92 (1947).

The author extends the result of A. G. Walker [same J. **19**, 106–107 (1944); these Rev. **6**, 263] that any real number z_0 is a zero of an integral function represented by a power series with real rational coefficients. He shows that the theorem still holds when z_0 is complex and furthermore that if z is complex with $|z|<1$ then z is a zero of a function whose power series representation has real integral coefficients and is convergent for $|z|<1$. *M. S. Robertson.*

Krein, M. A contribution to the theory of entire functions of exponential type. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **11**, 309–326 (1947). (Russian. English summary)

A function $f(z)$ is said to be of class N in a region if $\log^+|f(z)|$ has a harmonic majorant there. The author's principal results concern entire functions belonging to class N in half planes. In the first place he shows that, if $f(z)\epsilon N$ in the upper half plane, there is a real number μ such that $\limsup_{r\rightarrow\infty} r^{-1}\log|f(re^{i\varphi})|=\mu\sin\varphi$, $0<\varphi<\pi$, uniformly in any closed interior angle, while in any angle in which $f(z)$ has no zeros, $\lim r^{-1}\log|f(re^{i\varphi})|=\mu\sin\varphi$. He next shows that a necessary and sufficient condition for an entire function $f(z)$ to belong to N in both the upper and lower half planes is that $f(z)$ is of exponential type and satisfies $\int_{-\infty}^{\infty}(1+x^2)^{-1}\log^+|f(x)|dx<\infty$.

Special subclasses of N are (1) the class of $f(z)$ with real simple zeros α_k, such that $1/f(z)=P(z)+z^p\sum_1^\infty c_k/(z-\alpha_k)$, with $\sum|c_k|<\infty$, $P(z)$ a polynomial; (2) the class of $f(z)$ with simple zeros α_k such that $\sum\mathfrak{J}(\alpha_k)/|\alpha_k|^2<\infty$ and $1/f(z)=\sum_{k=1}^\infty c_k/(z-\alpha_k)$, $\sum|c_k|/|\alpha_k|<\infty$.

The results are obtained by mapping a half plane on a circle, so that known results on functions of class N in a circle can be applied [Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, 1936, chap. 7; Smirnov, J. Soc. Phys.-Math. Léningrade [Žurnal Leningradskogo Fiz.-Mat. Obščestva] **2**, no. 2, 22–37 (1928)]. An auxiliary theorem proved in the paper is that, if $F(z)\epsilon N$ in $|z|<1$, and Γ is a path going to $z=1$ inside $|z|<1$ and making a positive angle θ with the real axis at $z=1$, then $\limsup_{z\rightarrow 1}|1-z|\log|F(z)|=d\cos\theta$, where d does not depend on Γ. *R. P. Boas, Jr.* (Providence, R. I.).

Bernšteĭn, S. N. On limiting dependences between constants of the theory of best approximation. Doklady Akad. Nauk SSSR (N.S.) **57**, 3–5 (1947). (Russian)

Consider the class of functions $f(x)$ satisfying

$$|f^{(k)}(x+h)-f^{(k)}(x)|\leqq Mh^\alpha, \quad -\infty<x<\infty,$$

where $0<\alpha\leqq 1$. Write $A_p[f(x)]$ for the best approximation to $f(x)$ by entire functions of exponential type p, $E_n[f(x)]$ for the best approximation on $(-1, 1)$ to $f(x)$ by polynomials of degree n and $E_n^*[f(x)]$ for the best approximation to $f(x)$ by trigonometric polynomials of order n (if $f(x)$ is periodic with period 2π). It is easy to show that $A_p[f(x)]\leqq Mp^{-k-\alpha}c_{k,\alpha}$, where $c_{k,\alpha}$ is independent of p; let $E_n[f(x)]\leqq Mn^{-k-\alpha}c_{k,\alpha,n}$ and $E_n^*[f(x)]\leqq Mn^{-k-\alpha}d_{k,\alpha,n}$, where

in each case we take the minimum constant on the right. The author proves that $\lim_{n\to\infty} d_{k,\alpha,n} = \lim_{n\to\infty} c_{k,\alpha,n} = c_{k,\alpha}$. In particular, it follows from known results [Akhiezer, Lekcii po Teorii Aproksimacii, Kharkov, 1940; these Rev. **3**, 234] on $d_{k,1,n}$ that $c_{k,1} = 4\pi^{-1}\sum_{l=0}^{\infty}(-1)^{kl}(2l+1)^{-k-2} \leqq \pi/2$, a result which is used in the note reviewed below. The results depend on connections among the various kinds of best approximation, given by the author in earlier notes [C. R. (Doklady) Akad. Sci. URSS (N.S.) **54**, 103–108, 475–478 (1946); these Rev. **8**, 373, 509]. *R. P. Boas, Jr.*

Bernštein, S. N. On the best approximation to analytic functions with the aid of entire functions of finite degree. Doklady Akad. Nauk SSSR (N.S.) **56**, 891–894 (1947). (Russian)

The author gives some applications of theorem 7 of a recent note [second reference in the preceding review] in connection with a theorem of Akhiezer [reference in the preceding review] and other theorems of his own [preceding review and C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 487–490 (1946); these Rev. **8**, 20]. Notations: $E_n[f(x), a]$ is the best approximation to $f(x)$ by polynomials of degree not exceeding n on $(-a, a)$; $A_p[f(x)]$ is the best approximation to $f(x)$ on $(-\infty, \infty)$ by entire functions of exponential type p. The theorem of Akhiezer is as follows. If $f(z)$ is regular in the ellipse C with foci ± 1 and semi-minor axis b, and $|\Re f(z)| \leqq M$ there, then $E_n[f(x), 1] \leqq 8\pi^{-1}M\varphi\{[b+(b^2+1)^{\frac{1}{2}}]^n\}$, where

$$\varphi(q) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2\pi+1} \frac{1}{q^{2m+1}+q^{-(2m+1)}} < q^{-1}, \qquad q>1.$$

The results of the present note are as follows. (1) The condition $\limsup_{p\to\infty} A_p[f(x)]^{1/p} \leqq e^{-b}$ is necessary and sufficient for $f(x)$, bounded on $(-\infty, \infty)$, to satisfy

$$\limsup_{n\to\infty} \max_{x} \{|f^{(n)}(x)|/n!\}^{1/n} \leqq 1/b.$$

(2) A necessary and sufficient condition that $h_0 > 1$ should be the upper bound of numbers h for which

$$\lim_{p\to\infty} A_p[f(x)]^{p^{-h}} = 0$$

is that the bounded function $f(x)$ should be entire and that $\rho_0 = h_0/(h_0-1)$ should be the lower bound of numbers ρ for which $\lim_{b\to\infty} b^{-\rho}F(b) = 0$, where $F(b) = \sup |f(x+iy)|$ for $-b<y<b$ and $-\infty<x<\infty$. (3) A necessary and sufficient condition for the existence of $h>0$ such that $\limsup_{p\to\infty} A_p[f(x)]^{p^{-h}} = e^{-N}$, $0<N\leqq\infty$, is that the bounded function $f(x)$ should belong to class C^∞ and that

$$\limsup_{m\to\infty} \max_{-\infty<x<\infty} \{|f^{(mh)}(x)|/\Gamma(m+1)\}^{1/m} = 1/N.$$

R. P. Boas, Jr. (Providence, R. I.).

Shen, Yu-Cheng. Interpolation to some classes of analytic functions by functions with pre-assigned poles. Trans. Amer. Math. Soc. **62**, 338–356 (1947).

Denoting by S_α the class of functions $f(z)$ which are analytic for $|z|<1$ and such that

$$\mathfrak{M}_\alpha(f; R) = (\alpha-1)\pi^{-1}\int_0^R\int_0^{2\pi}(1-r^2)^{\alpha-2}|f(re^{i\Theta})|^2 r\,d\Theta\,dr \qquad (\alpha>1)$$

is bounded for $0<R<1$, the author deals with interpolation to functions of S_α at the points $a_{n,k}$ ($k=1, \cdots, n$; $n=1, 2, \cdots$) having no limit point inside $|z|=1$, by rational functions $f_n(z)$ of the form $\sum_{k=1}^{n} A_{n,k}(1-\bar{a}_{n,k}z)^{-\alpha}$. The limit-class S_1 is the Riesz class H_2, dealt with by J. L. Walsh [Interpolation and Approximation by Rational Functions in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 20, New York, 1935; cf. chaps. 8–10]. Some results for S_2 were proved by the author before [same Trans. **60**, 12–21 (1946); these Rev. **8**, 20]. Transforming $\{a_{n,k}\}$ into an "equivalent" sequence $\{b_{n,k}\}$ by

$$\text{(A)} \qquad \zeta = \lambda(z-c)(1-\bar{c}z)^{-1} \qquad (|\lambda|=1,\ |c|<1),$$

he proves now: in order that, for any $f(z)\varepsilon S_\alpha$, $f_n(z)\to f(z)$ uniformly for $|z|\leqq r$ whenever $r<1$, it is sufficient that there exists a sequence $\{b_{n,k}\}$ such that $n^{\alpha-1}\prod_{k=1}^{n}|b_{n,k}|^2\to 0$ as $n\to\infty$; it is necessary that $\prod_{k=1}^{n}|a_{n,k}|\to 0$; the latter condition is also sufficient if, for $n\geqq N$, the $b_{n,k}$ all lie on a diameter of $|z|=1$ ($1<\alpha<\infty$; the restriction $\alpha\leqq 2$ is removed in the addendum, § 10). The proofs are based on the formula

$$f(z)-f_n(z) = (\alpha-1)\pi^{-1}\int_{|t|<1}\int(1-|t|^2)^{\alpha-2}f(t)r_n(z;\bar{t})\,dS;$$

$r_n(z, \bar{t})$ is represented as the quotient of two determinants; $(1-z\bar{t})^\alpha r_n(z, \bar{t})$ is invariant under (A). *H. Kober.*

Boas, R. P., Jr. Correction: "Fundamental sets of entire functions." Ann. of Math. (2) **48**, 1095 (1947).

The paper appeared in the same Ann. (2) **47**, 21–32 (1946); these Rev. **7**, 425.

Wittich, H. Über die Wachstumsordnung einer ganzen transzendenten Funktion. Math. Z. **51**, 1–16 (1947).

The mapping of a simply connected Riemann surface with p logarithmic and a finite number of algebraic branch-points onto the plane yields a meromorphic function of order $p/2$. In the case of infinitely many algebraic branch-points $p/2$ is still a lower bound of the order, but it was not known whether the order can be effectively raised by introduction of algebraic branch-points. By explicit construction of an example the author shows that this is so. In this example the defects of the function are not changed. *L. Ahlfors* (Cambridge, Mass.).

Walsh, J. L. The location of the critical points of simply and doubly periodic functions. Duke Math. J. **14**, 575–586 (1947).

Elementare Betrachtungen über die Lage der kritischen Punkte, insbesondere der Nullstellen und Pole von einfach und zweifach periodischen Funktionen. *P. J. Myrberg.*

Epstein, Bernard. Some inequalities relating to conformal mapping upon canonical slit-domains. Bull. Amer. Math. Soc. **53**, 813–819 (1947).

Let D be an n-connected domain in the z-plane containing the point $z=\infty$ and let $\zeta = s_\theta(z) = z + a_\theta/z + \cdots$ be the (uniquely determined) function mapping D in a schlicht manner onto the ζ-plane bounded by rectilinear slits making the angle θ with the positive axis. It was known previously [H. Grunsky, Schr. Math. Sem. Univ. Berlin **1**, 95–140 (1932); M. Schiffer, Duke Math. J. **10**, 209–216 (1943); these Rev. **4**, 271] that $S = a_0 - a_{\pi/2}$ (the "span" of D) is positive and subject to the inequality $S \geqq 2A/\pi$, where A is the outer measure of D, i.e., the greatest lower bound of the total area enclosed by a set of analytic curves surrounding the boundary continua of D. The author supplements this result by the sharp inequality

$$\Re\{a_\theta e^{-2i\theta}\} \geqq A/(2\pi) + |a_\theta|^2/S.$$

The proof is based on the theory of orthonormal systems of analytic functions. [See S. Bergman, Amer. J. Math. **68**, 20–28 (1946); these Rev. **7**, 286.] By a simpler method, the author proves the weaker inequality $\Re\{a_0 e^{-2i\theta}\} \geqq A/(2\pi)$ which, however, is also "best possible" in the sense that the constant $(2\pi)^{-1}$ cannot be improved. *Z. Nehari.*

Luzin, N. N. On the localization of the principle of finite area. Doklady Akad. Nauk SSSR (N.S.) **56**, 447–450 (1947). (Russian)

The author localizes Fejér's theorem that if for a function $f(z)=\sum_{n=0}^{\infty} a_n z^n$, analytic in $|z|<1$, (1) $\sum_{n=0}^{\infty} n|a_n|^2$ converges, then the series $\sum_{n=0}^{\infty} a_n z^n$ converges at all points $e^{i\theta}$ of $|z|=1$ for which $\lim_{r\to 1} f(re^{i\theta})$ exists, the convergence being uniform on any set for which $\lim f(z)$ is uniform. The convergence of series (1) means geometrically that the area of the Riemann surface on which $f(z)$ maps the circle $|z|<1$ is finite. The localization of this theorem is as follows. If $f(z)=\sum_{n=0}^{\infty} a_n z^n$, analytic in $|z|<1$, maps a fixed sector $A \leqq \arg z \leqq B$, $|z|<1$, on a Riemann surface of finite area and if $\lim_{n\to\infty} a_n=0$, then $\sum_{n=0}^{\infty} a_n z^n$ converges at all points $e^{i\theta}$ of any arc $A<\alpha\leqq\theta\leqq\beta<B$ for which $\lim_{r\to 1} f(re^{i\theta})$ exists, the convergence being uniform on any subset of the arc for which $\lim f(z)$ is uniform. By an extension of Fatou's theorem $\sum_{n=0}^{\infty} a_n z^n$, $z=re^{i\theta}$, must, therefore, converge at all points of the arc $\alpha\leqq\theta\leqq\beta$, $r=1$, except perhaps at a set of measure zero. [Two proofs of the above result have been given by A. Zygmund, Bull. Sém. Math. Univ. Wilno **2**, 3–12 (1939); these Rev. **1**, 9.] Some points in Luzin's proof remain unclear to the reviewer. In referring to a certain result concerning the convergence of power series on their circle of convergence, the author ascribes the result to Fatou, whereas the second part of the theorem is due to M. Riesz [J. Reine Angew. Math. **140**, 89–99 (1911)]. The author concludes with some conjectures and problems concerning bounded analytic functions. *W. Seidel.*

Richards, Paul I. A special class of functions with positive real part in a half-plane. Duke Math. J. **14**, 777–786 (1947).

Let S denote the class of functions $f(z)=U(x,y)+iV(x,y)$, $z=x+iy$, for which (1) $f(z)$ is analytic and single-valued in $x>0$, (2) $f(z)$ has no singularities other than poles on $x=0$, (3) $f(z)$ is real on the real axis, (4) $U(x,y)\geqq 0$ for $x\geqq 0$. The author gives a unified exposition of the properties of the functions of this class. Some of these properties are well known, but because of condition (2) certain new results are included. A few may be mentioned here.

(I) If $V(0,y)\equiv 0$ then $f(z)$ is a real nonnegative constant. (II) If $f(z)/f(1)\not\equiv z$ or $1/z$ then $\{f(z)-zf(1)\}/\{f(1)-zf(z)\}\in S$. (III) If $|f(z)|<B$ on $x=0$, then $|f(z)|<B^2/f(1)$ for $x\geqq 0$. (IV) For $x>0$,

$$f(z)=Az+\frac{k_0}{z}+2z\sum_1^{\infty}\frac{k_n}{z^2+b_n^2}+\frac{2z}{\pi}\int_0^{\infty}\frac{U(0,t)dt}{z^2+t^2},$$

where $\lim_{|z|\to\infty} f(z)/z=A\geqq 0$ in $|\arg z|\leqq\theta<\pi/2$, $z=\pm ib_n$ are the poles of $f(z)$ on $x=0$ (and possibly $z=0$), k_n are the residues of these poles. (V) If $f(z)$ is analytic on the (finite) imaginary axis, and if $\lim_{|z|\to\infty} f(z)/z=0$ in $|\arg z|<\pi/2$, then

$$V(0,c)=2c\pi^{-1}\int_0^{\infty}\frac{U(0,t)-U(0,c)}{t^2-c^2}dt.$$

(VI) If $U(0,y)\equiv 0$, $f(0)=0$, then

$$f(z)=zf'(0)\prod_1^{\infty}\frac{1+(z/a_n)^2}{1+(z/b_n)^2},$$

where $\pm ia_n$ are the zeros and $\pm ib_n$ the poles of $f(z)$.

M. S. Robertson (New Brunswick, N. J.).

Bergman, Stefan. Functions satisfying certain partial differential equations of elliptic type and their representation. Duke Math. J. **14**, 349–366 (1947).

Some methods and theorems from the theory of analytic functions are generalized for pairs of real functions (Φ, Ψ) which in a (multiply connected) domain B satisfy a pair of equations

$$(1)\qquad \Phi_x=l(x,y)\Psi_y,\quad \Phi_y=-l(x,y)\Psi_x,$$

which of course are identical with the Cauchy-Riemann equations if $l\equiv 1$. Among these pairs of functions integrals of the first, second and third kind are defined which correspond to Abelian integrals in the classical theory of algebraic functions. Elimination of Ψ from (1) yields

$$(2)\qquad \Phi_{xx}+\Phi_{yy}-(\log l)_x\Phi_x-(\log l)_y\Phi_y\equiv L(\Phi)=0.$$

For those functions Φ which satisfy (2) in a domain B, a new definition of orthogonality is given, which depends on L. It is shown that for each L and each domain B a complete orthonormal system exists. Orthonormal systems are finally used in order to find an integral representation for integrals of the first kind. *W. Nef* (Fribourg).

Fourier Series and Generalizations, Integral Transforms

Salem, R., and Zygmund, A. On lacunary trigonometric series. Proc. Nat. Acad. Sci. U. S. A. **33**, 333–338 (1947).

A propos des séries trigonométriques lacunaires du type $\sum_{k=1}^{\infty}(a_k\cos n_k x+b_k\sin n_k x)$ où les n_k sont des nombres positifs tels que, quel que soit k, $n_{k+1}/n_k>q>1$, q étant une constante donnée, les auteurs résolvent le problème suivant, déjà abordé par d'autres auteurs [cf., par exemple, Ferrand et Fortet, C. R. Acad. Sci. Paris **224**, 516–518 (1947); ces Rev. **8**, 391, où l'on trouvera d'autres références]: posons $c_k^2=a_k^2+b_k^2$, $C_N^2=\frac{1}{2}\sum_{k=1}^{N} c_k^2$, $S_N(x)=\sum_{k=1}^{N}(a_k\cos n_k x+b_k\sin n_k x)$; désignons par E_0 l'intervalle $(0, 2\pi)$, par E un sous-ensemble mesurable quelconque de E_0 de mesure $|E|>0$, par $E_N(y)$, de mesure $|E_N(y)|$, l'ensemble des x de E pour lesquels $S_N(x)/C_N\leqq y$, par $F_N(E;y)$ la fonction de répartition (en y) égale à $|E_N(y)|/|E|$; lorsque, E restant fixe, $N\to+\infty$, dans quelles conditions $F_N(E;y)$ tend elle vers une fonction de répartition de Laplace (nécessairement d'espérance mathématique 0 et d'écart moyen quadratique 1)?

Les auteurs démontrent que: (a) pour que $F_N(E_0;y)$ tende vers une fonction de répartition $F(y)$ telle que $0<F(y)<1$ pour tout y fini, il faut que (1) $c_N/C_N\to 0$; (b) si la condition (1) est réalisée et si $C_N\to+\infty$, $F_N(E;y)$ tend, quel que soit E, vers une fonction de répartition de Laplace. Les auteurs complètent ces résultats, et en particulier les étendent aux séries lacunaires complexes du type $\sum_k c_k e^{in_k x}$, et signalent la possibilité d'appliquer leur méthode à d'autres problèmes du même ordre. *R. Fortet* (Caen).

***Broman, Arne. On two classes of trigonometrical series.** Thesis, University of Uppsala, Uppsala, 1947. 51 pp.

Let S_α, $0<\alpha\leqq 1$, be the class of all trigonometric series (T) $\frac{1}{2}a_0+\sum(a_n\cos nx+b_n\sin nx)$ such that $\sum n^\alpha(a_n^2+b_n^2)$ converges. Obviously every $T\epsilon S_\alpha$ is the Fourier series of a Lebesgue integrable function f. The class of series T with $\sum n^{-\alpha}(a_n^2+b_n^2)<\infty$ will be denoted by T_α. Beurling [Acta Math. **72**, 1–13 (1939); these Rev. **1**, 226] proved the following result. (1) If the Fourier series of a function f is of the class S_1, then (a) it converges to a finite limit outside a set whose logarithmic capacity is zero, (b) the primitive of f has a finite derivative outside a set of logarithmic capacity zero. He also stated without proof that, if the Fourier series of f belongs to $S_{\alpha-\epsilon}$ $(0<\alpha<1)$ for every $\epsilon>0$, then the Poisson integral $f(r,\theta)$ of f has a finite radial limit outside a set whose capacitary dimension is $1-\alpha$. The latter result was proved in a slightly more general form by Salem and Zygmund [Trans. Amer. Math. Soc. **59**, 23–41 (1946); these Rev. **7**, 434] who showed that (2a): if $T\epsilon S_\alpha$ $(0<\alpha<1)$, then T converges to a finite sum outside a set of capacity $1-\alpha$. In the present paper it is shown that (2b): under the assumptions of (2a), outside a set whose capacity of order $1-\alpha$ is zero, the primitive function of f has a finite derivative equal to the sum of the series. Moreover, (3) let the Fourier series of f be of the class S_α, and let $F(\theta)=\int_0^1|f_r(r,\theta)|dr$; then $F(\theta)$ is finite outside a set whose capacity of order $1-\alpha$ is zero, and the Fourier series of F is also of the class S_α. (4) A necessary and sufficient condition for a closed set to be a set of uniqueness (the sum of the series being taken in the Abel-Poisson sense), with respect to the class T_α, is that the capacity of order $1-\alpha$ of the set is zero. For $\alpha=1$ (when capacity is meant as the logarithmic capacity), results (3) and (4) had been obtained previously by Beurling. *A. Zygmund* (Chicago, Ill.).

Wang, Fu Traing. A remark on (C) summability of Fourier series. J. London Math. Soc. **22**, 40–47 (1947).

Let $f(x)$ be L-integrable and of period 2π and let $A_n(x)$ be the general term of the Fourier series of f,

$$\varphi(t)=\tfrac{1}{2}\{f(x+t)+f(x-t)-2s\},$$

$$\varphi_\alpha(t)=\{1/\Gamma(\alpha)\}\int_0^t(t-u)^{\alpha-1}\varphi(u)du \qquad (t>0,\ \alpha>0).$$

Hardy and Littlewood showed [J. London Math. Soc. **7**, 252–256 (1932)] that if (i) $\varphi(t)=o\{1/(\log t^{-1})\}$, (ii) $A_n(x)\geqq -Kn^{-\delta}$ $(\delta>0)$, then the Fourier series of f converges to s at the point x. They also showed that the conclusion does not hold if condition (ii) is omitted. The author now proves that, if $\varphi_\alpha(t)=o\{t^\alpha/(\log t^{-1})\}$, the Fourier series of f is summable (C,α) at x to sum s, without any condition corresponding to (ii) being imposed. *A. Zygmund.*

Fejes, László. Über die Fouriersche Reihe der Abkühlung. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **61**, 478–495 (1942). (Hungarian. German summary)

The series in question has the form

$$f(x)\sim\sum(a_\nu\cos\mu_\nu x+b_\nu\sin\mu_\nu x),$$

where μ_ν are the nonnegative roots of $z+h\tan az=0$ $(h>0,$ $a>0)$ [cf. Fejes, Acta Univ. Szeged. Sect. Sci. Math. **11**, 28–36 (1946); these Rev. **8**, 263]; the coefficients are determined with reference to a fundamental interval (x_0,x_1) of length $2a$. The author shows that the series converges or diverges at x if and only if it converges at $x+2ma$, $m=\pm1$, $\pm2,\cdots$. If $s_n(x)$ is the nth partial sum of the series, $s_n(x)+s_n(x+2a)$ converges for $x_0\leqq x\leqq x_1$ to a limit $F(x)$, and $f(x)=F(x)+h^{-1}F'(x)$ almost everywhere. *R. P. Boas, Jr.* (Providence, R. I.).

Tagamlitzki, Yaroslav. Sur l'équation intégrale de Stieltjes. C. R. Acad. Sci. Paris **225**, 976–978 (1947).

It is stated that a set of conditions necessary and sufficient for a function $f(x)$ analytic at $x=a>0$ to be representable by a Stieltjes transform $\int_0^\infty(x+t)^{-1}d\alpha(t)$, $\alpha(t)$ nondecreasing, is

$$\text{(i)}\qquad (-1)^k\frac{d^{k+n}}{dx^{k+n}}x^nf(x)\geqq 0,$$

$$\text{(ii)}\qquad \lim_{\nu\to\infty}x^\nu f^{(\nu)}(x)/\nu!=0,$$

$$\text{(iii)}\qquad \sum_{\nu=0}^\infty\frac{d^\nu}{dx^\nu}x^\nu f(x)<\infty$$

at $x=a$; $k=0,1,\cdots$; $n=0,1,\cdots$. The example $f(x)=1/x$, $a=1$ shows (ii) to be incorrect, and the example $f(x)=1/(x+1)$ contradicts (iii). It is not observed that the correct condition (i) already occurs in the work of Widder [Trans. Amer. Math. Soc. **39**, 244–298 (1936), in particular, p. 285]. It is stated further that a function suitably "dominated" by a Stieltjes transform is a Stieltjes transform. *H. Pollard* (Ithaca, N. Y.).

Polynomials, Polynomial Approximations

Rényi, A. On the minimal number of terms of the square of a polynomial. Hungarica Acta Math. **1**, 30–34 (1947).

Denote by $Q(n)$ the minimum number of terms of the square of a polynomial containing n terms. Put $g(n)=Q(n)/n$. Rédei raised the question of investigating $Q(n)$. Kalmár, Rédei and the author proved that $\liminf g(n)=0$. The author proves that $n^{-1}\{g(1)+\cdots+g(n)\}\to 0$. The proof follows easily from the following facts: $g(n\cdot m)\leqq g(n)\cdot g(m)$; $g(n)<3/2$; $g(4n+1)<28/29$. The author conjectures that $\lim g(n)=0$. *P. Erdös* (Syracuse, N. Y.).

Gaspar Teixeira, José. Sur une certaine classe de polynomes à coefficients complexes. Anais Fac. Ci. Pôrto **29**, 81–88 (1944).

According to the well-known theorem of Cauchy, all the zeros of a polynomial $f(z)=a_0+a_1z+\cdots+a_nz^n$ lie in the circle C: $|z|=r$, where r is the positive root of the equation $|a_0|+|a_1|z+\cdots+|a_{n-1}|z^{n-1}=|a_n|z^n$. In the present paper it is proved that for $f(z)$ to have a zero on C it is necessary and sufficient that $\arg(a_j/a_{j+1})=(1/n)\arg(a_0/a_n)+(\pi/n)$, for $j=0,1,\cdots,n-2$. The paper also considers polynomials $F(z)$ whose zeros are symmetric in some line L through the origin. If K: $|z|=R$ is the circle of univalence for $F(z)$, then $F(z)$ has either exactly two zeros on K and no zeros in K or exactly one zero in K on L. The latter theorem is an immediate consequence of the more general theorems of Montel on the univalence of analytic functions. *M. Marden.*

de Bruijn, N. G., and Springer, T. A. On the zeros of composition-polynomials. Nederl. Akad. Wetensch., Proc. **50**, 895–903 = Indagationes Math. **9**, 406–414 (1947).

Let $C(n,j)$ denote the binomial coefficient and let

$$A(z)=\sum C(n,j)a_jz^j=a_n\prod(z-\alpha_j),$$
$$B(z)=\sum C(n,j)b_jz^j=b_n\prod(z-\beta_j),$$
$$AB(z)=\sum C(n,j)a_jb_jz^j=a_nb_n\prod(z-\gamma_j),$$
$$\{A,B\}=\sum(-1)^jC(n,j)a_jb_{n-j},$$

where $j=0, 1, \cdots, n$ in the above sums and $j=1, 2, \cdots, n$ in all products and in all subsequent sums. The authors show that, if $\psi(z, L)$ denotes the distance of point z from a line L, then

$$\sum\{\psi(\gamma_j, L)-\psi(0, L)\}\leqq(b_{n-1}/b_n)\sum\{\psi(\alpha_j, L)-\psi(0, L)\}$$

provided that for all j either $-1\leqq\beta_j\leqq0$, or $\psi(0, L)=0$ and $\beta_j\leqq0$. Their proof is based upon some relations which are established by use of Grace's apolarity theorem.

Furthermore, the authors show that, if $\varphi(t)$ denotes $\max(t, 1)$ and if $r_1>0$ and $r_2>0$, then

$$\prod\varphi(r_1r_2/|\gamma_j|)\leqq\prod\varphi(r_1/|\alpha_j|)\varphi(r_2/|\beta_j|).$$

This theorem, which is established by use of the Jensen formula, leads to a number of relations. For instance, if $\psi(x)$ is a convex function and if $\lambda>0$ and $\mu>0$, then

$$(\lambda+\mu)\sum\psi\{(\log|\gamma_j|)/(\lambda+\mu)\}$$
$$\leqq\lambda\sum\psi\{(\log|\alpha_j|)/\lambda\}+\mu\sum\psi\{(\log|\beta_j|)/\mu\}.$$

For the choice $B(z)\equiv nz(1+z)^{n-1}$, $AB(z)=zA'(z)$ making γ_j equal to the zeros δ_j of the derivative of $A(z)$, and thus yielding the relation

$$\sum_{j=1}^{n-1}\psi(\log|\delta_j|)\leqq\sum_{j=1}^{n}\psi(\log|\alpha_j|)$$

when $\psi(x)$ is further assumed to be nondecreasing with $\psi(-\infty)\geqq0$. The latter inequality generalizes previous results [de Bruijn, same Proc. **49**, 1037–1044 = Indagationes Math. **8**, 635–642 (1946); de Bruijn and Springer, same Proc. **50**, 264–270 = Indagationes Math. **9**, 458–464 (1947); these Rev. **8**, 377; **9**, 30]. *M. Marden.*

Leja, F. Sur les polynomes de Tchebycheff et la fonction de Green. Ann. Soc. Polon. Math. **19** (1946), 1–6 (1947).

Let E be a closed set with a positive transfinite diameter d in the complex z-plane, $T_n(z)=z^n+\cdots$ the associated T-polynomial of degree n (for which $\max|T_n(z)|$ on E is a minimum). Let D be the component of the complementary set of E which contains $z=\infty$. It is shown that $\lim|T_n(z)|^{1/n}$ exists uniformly on every closed set in D not containing any point of accumulation of the zeros of $T_n(z)$. The limit is $d\exp(G)$, where G is the Green's function of D with respect to $z=\infty$. The example of two equal intervals situated on the same line is used as illustration. *G. Szegö.*

Special Functions

***Magnus, Wilhelm, und Oberhettinger, Fritz. Formeln und Sätze für die speziellen Funktionen der mathematischen Physik.** Springer-Verlag, Berlin, 1943. viii+172 pp.

This book fills a long-felt gap in the literature, and it is certain to be welcomed by every practical mathematician. It gives a brief survey of the principal properties of the more important special functions of applied mathematics together with a very judiciously selected list of formulae involving these functions.

The scope of the work will be best seen from the table of contents. Chapter I: The gamma function. Chapter II: The hypergeometric function. Chapter III: Bessel functions (with an appendix on Mathieu functions). Chapter IV: Legendre functions (including conal and toroidal harmonics). Chapter V: Orthogonal polynomials (Chebyshev, Hermite, Jacobi, and Laguerre polynomials). Chapter VI: The confluent hypergeometric function and its special cases. Chapter VII: Theta functions, elliptic functions and integrals. Chapter VIII: Integral transforms and inversion of definite integrals (Fourier, Laplace, and Hankel transforms, all three with fairly extensive lists of pairs of transforms, Mellin, Gauss transforms, and various integral equations of the first kind). Chapter IX: Curvilinear coordinates (transformation of differential operators to practically all the well-known systems of curvilinear orthogonal coordinates, separation of the potential, wave, and diffusion equations in these coordinates).

There are a certain number of misprints; one misses some topics, and would have others treated in a slightly different manner; but it would be ungracious to complain about details when on the contrary the authors must be congratulated on the excellent manner in which they carried out their task under difficult circumstances. A revised edition (incorporating additions as well as corrections) is in preparation. *A. Erdélyi* (Pasadena, Calif.).

***MacRobert, T. M. Spherical Harmonics. An Elementary Treatise on Harmonic Functions with Applications.** 2d ed. Methuen & Co., Ltd., London, 1947. xv+372 pp. 22/6.

This work contains a great deal more than its title would seem to promise. It is a very useful text-book on special functions, and an introduction to their application to partial differential equations of mathematical physics. The treatment is at the level of a course in advanced calculus; accordingly no contour integration methods are used, and all variables and parameters are real, except in the last two chapters in which the variable is complex.

The scope of the book will be best seen from the table of contents. I. Fourier series. II. Conduction of heat. III. Transverse vibrations of stretched strings. IV. Spherical harmonics; the hypergeometric function. V. The Legendre polynomials. VI. The Legendre functions. VII. The associated Legendre functions of integral order. VIII. Applications of Legendre coefficients to potential theory. IX. Potentials of spherical shells, spheres, and spheroids. X. Applications to electrostatics. XI. Ellipsoids of revolution. XII. Eccentric spheres. XIII. Clerk Maxwell's theory of spherical harmonics. XIV. Bessel functions. XV. Asymptotic expansions and Fourier-Bessel expansions. XVI. Applications of Bessel functions. XVII. The hypergeometric function. XVIII. Associated Legendre functions of general order. A large number of examples, some with solutions and many with hints for the solution, complement the text.

The last two chapters, on the hypergeometric function and on Legendre functions of general (not necessarily integer) real degree and order and a set of miscellaneous examples are new in the present second edition and enlarge the scope of the work in a welcome manner. *A. Erdélyi* (Pasadena, Calif.).

MacRobert, T. M. Some applications of contour integration. Philos. Mag. (7) **38**, 45–51 (1947).

The object of this paper is to give comparatively simple proofs by contour integration of some known theorems. Section 2 contains a discussion of the analytic continuation of the hypergeometric function. In the same way the author derives in § 3 the following formula of Hobson for the asso-

ciated Legendre functions of the second kind:

$$Q_n^m(z)=\Gamma(m-n)\Gamma(m+n+1)\{2\Gamma(m+1)\}^{-1}$$
$$\times\left\{\begin{array}{l} e^{\mp m\pi i/2}\left(\frac{1+z}{1-z}\right)^{m/2}F\left(\begin{matrix}-n,\ n+1\\ m+1\end{matrix};(1+z)/2\right)\\ -e^{\mp(n-\frac{1}{2}m)\pi i}\left(\frac{1-z}{1+z}\right)^{m/2}F\left(\begin{matrix}-n,\ n+1\\ m+1\end{matrix};(1-z)/2\right)\end{array}\right\},$$

where the upper signs are taken for $\Im(z)>0$, the lower for $\Im(z)<0$. The asymptotic expansion of Kummer's function

$$F(\alpha;\rho;iz)=\frac{\Gamma(\rho)}{\Gamma(\alpha)\Gamma(\rho-\alpha)}\int_0^1 e^{iz\xi}\xi^{\alpha-1}(1-\xi)^{\rho-\alpha-1}d\xi,$$

$\Re(\rho)>\Re(\alpha)>0$, is obtained by means of the E-function,

$$E(\alpha,\beta::z)=\Gamma(\alpha)\int_0^\infty e^{-\lambda}\lambda^{\beta-1}(1+\lambda/z)^{-\alpha}d\lambda,$$

$\Re(\beta)>0$, in the form

$$\frac{\Gamma(\alpha)\Gamma(\rho-\alpha)}{\Gamma(\rho)}F(\alpha;\rho;z)=\frac{e^z z^{\alpha-\rho}E(1-\alpha,\rho-\alpha::ze^{-i\pi})}{\Gamma(1-\alpha)}$$
$$+\frac{e^{i\pi\alpha}z^{-\alpha}E(\alpha,\alpha-\rho+1::z)}{\Gamma(\alpha-\rho+1)},$$

$-\pi/2<\operatorname{amp} z<3\pi/2$, and an analogous formula for $-3\pi/2<\operatorname{amp} z<\pi/2$. The cited formula is equivalent to Whittaker's formula for $M_{k,m}$ [Whittaker and Watson, A Course of Modern Analysis, 4th ed., Cambridge University Press, 1927, p. 346, with correction of an error in sign]. In a former paper [Proc. Roy. Soc. Edinburgh 51, 116–126 (1931)] Fourier's integral theorem

$$\int_{-\infty}^\infty e^{-\alpha_\lambda r}\left(\int_p^q e^{\alpha_\lambda\rho}\varphi(\rho)d\rho\right)d\lambda=\begin{cases}2\pi\varphi(r), & -\infty\leqq p<r<q\leqq\infty;\\ 0, & r<p \text{ or } r>q,\end{cases}$$

was established by contour integration for holomorphic functions. Now this proof is modified to include the boundary cases:

$$P\int_{-\infty}^\infty e^{-\alpha_\lambda r}\left(\int_p^q e^{\alpha_\lambda\rho}\varphi(\rho)d\rho\right)d\lambda=\pi\varphi(r),\quad r=p \text{ or } r=q.$$

In a similar way the Fourier-Bessel integral theorem is now stated as follows:

$$\int_0^\infty\left\{\int_p^q\varphi(\rho)J_n(\lambda\rho)\rho d\rho\right\}J_n(\lambda r)\lambda d\lambda$$
$$=\begin{cases}\varphi(r), & 0\leqq p<r<q\leqq\infty;\\ \frac{1}{2}\varphi(r), & 0<r=p;\ r=q>p;\\ 0, & 0<r<p \text{ or } r>q.\end{cases}$$

S. C. van Veen (Delft).

Shabde, N. G. Two integrals involving Legendre functions. Proc. Benares Math. Soc. (N.S.) 7, no. 2, 51–53 (1945).

The author evaluates

$$\int_{-1}^1(1+z)^pP_n(z)dz$$

and

$$\int_{-1}^1(1+z)^{m+n}P_m(z)P_n(z)dz,$$

where $P_n(z)$ is the Legendre function of the first kind, $p>-1$ and $m+n>-1$; m, n, p need not be integers.

A. Erdélyi (Pasadena, Calif.).

Fasenmyer, Mary Celine. Some generalized hypergeometric polynomials. Bull. Amer. Math. Soc. 53, 806–812 (1947).

Certain special sets of generalized hypergeometric polynomials

$$f_n(a_i;b_j;x)\equiv f_n(a_1,\cdots,a_p;b_1,\cdots,b_q;x)$$
$$\equiv{}_{p+2}F_{q+2}\left[\begin{matrix}-n,\ n+1,\ a_1,\ \cdots,\ a_p;\\ \frac{1}{2},\ \ 1,\ \ b_1,\ \cdots,\ b_q;\end{matrix}\ x\right]$$

containing among others Legendre's, Jacobi's and Bateman's polynomials, are considered here. It is shown that

$$f_n(a_i;b_j;x)=\frac{(-4x)^n(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_nn!}$$
$$\times{}_{q+3}F_{p+1}\left[\begin{matrix}-n,\ -n,\ \frac{1}{2}-n,\ 1-b_1-n,\ \cdots,\ 1-b_q-n;\\ -2n,\ \ 1-a_1-n,\ \cdots,\ 1-a_p-n;\end{matrix}\ (-1)^{q-p+1}/x\right]$$

in which $(a)_r=a(a+1)\cdots(a+r-1)$; $(a)_0=1$. A differential recurrence formula is

$$\frac{xd}{dx}[f_n(x)+f_{n-1}(x)]=n[f_n(x)-f_{n-1}(x)].$$

Pure recurrence relations, contiguous polynomial relations and integral relations are stated without proof. From the last kind we mention in particular

$$f_n(a_i;b_j;x)=\pi^{-\frac{1}{2}}\int_0^\infty y^{-\frac{1}{2}}e^{-y}f_n(a_i;\tfrac{1}{2},b_j;xy)dy.$$

For Bateman's $Z_n(t)=f_n(\frac{1}{2};1;t)$ it is easy to write

$$Z_n(t^2)=\pi^{-\frac{1}{2}}\int_0^\infty y^{-\frac{1}{2}}e^{-y}f_n(-;1;t^2y)dy$$
$$=\pi^{-\frac{1}{2}}\int_0^\infty e^{-\alpha^2/4}L_n(\alpha t)L_n(-\alpha t)d\alpha$$

($L_n(x)$ is a Laguerre polynomial) by Ramanujan's formula for the product of two ${}_1F_1$'s as a ${}_2F_3$ [Bailey, Generalised Hypergeometric Series, Cambridge University Press, 1935, p. 97].

S. C. van Veen (Delft).

Powell, John L. Recurrence formulas for Coulomb wave functions. Physical Rev. (2) 72, 626–627 (1947).

The Coulomb wave functions F_L and G_L are solutions of the differential equation

$$\frac{d^2F}{d\rho^2}+\left\{1-\frac{2\eta}{\rho}-\frac{L(L+1)}{\rho^2}\right\}F=0,\qquad L=1,2,\cdots,$$

which, for large ρ, have the asymptotic forms

$$F_L\sim\sin(\rho-\tfrac{1}{2}L\pi-\eta\ln 2\rho+\sigma_L),$$
$$G_L\sim\cos(\rho-\tfrac{1}{2}L\pi-\eta\ln 2\rho+\sigma_L),$$

$\sigma_L=\arg\Gamma(L+1+i\eta)$, where

$$F_L+iG_L=Y_L$$
$$=\frac{\rho^{L+1}}{i(e^{2\pi\eta}-1)(2L+1)!C_L}\int_D(z-i)^{L+i\eta}(z+i)^{L-i\eta}e^{i\rho z}dz,$$
$$C_L^2=\frac{2\pi\eta}{e^{2\pi\eta}-1}\,\frac{\prod_{m=1}^N(1+\eta^2/m^2)}{\prod_{m=0}^N(2m+1)^2},$$

and D is a contour which starts at $(-\infty-i)$, encircles the point $-i$ once in the positive sense, and returns to the starting point. The function Y_L (and also F_L and G_L) satis-

fies the relations

$$(L+1)\{L^2+\eta^2\}^{\frac{1}{2}}Y_{L-1}-(2L+1)\{\eta+\rho^{-1}L(L+1)\}Y_L +L\{(L+1)^2+\eta^2\}^{\frac{1}{2}}Y_{L+1}=0,$$
$$(L+1)Y_L'-\{\rho^{-1}(L+1)^2+\eta\}Y_L+[(L+1)^2+\eta^2]^{\frac{1}{2}}Y_{L+1}=0,$$
$$LY_L'+(\rho^{-1}L^2+\eta)Y_L-(L^2+\eta^2)^{\frac{1}{2}}Y_{L-1}=0$$

(the primes denote differentiation with respect to ρ).

S. C. van Veen (Delft).

Parodi, Maurice. Relations intégrales entre les fonctions $\nu(t)$ et $\mu(t, x)$ de Serge Colombo. Revue Sci. **85**, 360 (1947).

For the functions

$$\mu(t, x)=\int_0^\infty \frac{t^s s^x ds}{\Gamma(1+s)}, \quad \nu(t)=\mu(t, 0),$$

the author proves

$$\Gamma(1-x)\mu(t, x)=\int_0^\infty \mu(te^{-\beta}, 1)\beta^{-x}d\beta,$$

$$\Gamma(x)\nu(t)=\int_0^\infty \mu(te^{-\beta}, x)\beta^{x-1}d\beta.$$

[Reviewer's remark: both these relations are special cases of

$$\int_0^\infty \mu(te^{-\beta}, x)\beta^{y-1}d\beta=\Gamma(y)\mu(t, x-y).]$$

The results are generalised to similarly defined functions $\nu(t, n)$ and $\mu(t, x, n)$. *A. Erdélyi* (Pasadena, Calif.).

Humbert, Pierre. Images des fonctions de Mathieu. C. R. Acad. Sci. Paris **225**, 715–716 (1947).

There are many integral equations satisfied by, and integral relations between, Mathieu functions. By using such relations the author attempts to solve the problem of operational representations of Mathieu functions. In particular, from one pair of integral formulae for modified Mathieu functions he shows that [in the notation of N. W. McLachlan's Theory and Applications of Mathieu Functions, Oxford, 1947; these Rev. **9**, 31] the operational image of $\mathrm{Se}_n(\cosh^{-1}(1+t))$ is a numerical multiple of $p(p^2+4k^2)^{-\frac{1}{2}}e^p\, \mathrm{Gek}_n\, (\cosh^{-1} \frac{1}{2}ip/k)$. *A. Erdélyi.*

Campbell, Robert. Comportement asymptotique des fonctions de Mathieu associées pour des paramètres infiniment grands. C. R. Acad. Sci. Paris **225**, 371–373 (1947).

The author considers the solution of period 2π of the differential equation

$$y''+y\{a+\nu^2-\nu(\nu-1)\sec^2 x+k^2\sin^2 x\}=0$$

in the case when the parameters a, ν, k are infinite. The most interesting case arises if k and ν are infinite and of the same order. The author assumes k to be a function of ν, while a is to be determined so that a periodic solution as stated above arises. He supposes that

$$k^2=\alpha\nu^2+\beta\nu+\gamma+\delta\nu^{-1}+\cdots,$$
$$a=A\nu^2+B\nu+C+D\nu^{-1}+\cdots,$$

and assumes a corresponding asymptotic solution. Under proper conditions asymptotic solutions similar to Weber functions arise and the parameters may be determined from the condition that these Weber functions must be of integral order. By extending the solutions across the critical points the author obtains the behavior under the specified asymptotic conditions. *M. J. O. Strutt* (Eindhoven).

Differential Equations

Horvay, Gabriel, and Yuan, S. W. Stability of rotor blade flapping motion when the hinges are tilted. Generalization of the "rectangular ripple" method of solution. J. Aeronaut. Sci. **14**, 583–593 (1947).

It is established that the oscillations of a helicopter rotor blade obey the equation $\beta''+A\beta'+B\beta=E$, where A, B, and E are periodic functions in time. The essential question asks whether the solutions of the homogeneous equation are stable. Approximate solutions are found by replacing A and B by appropriately chosen piece-wise constant periodic functions. The stability of these solutions is computed over a wide range of the parameters which are included in A and B. It is concluded that the oscillations are stable under conventional flight conditions. The conditions under which forced resonant oscillations may occur are also discussed.

G. F. Carrier (Providence, R. I.).

Lourye, A. I. Investigation of the stability of motion of a dynamic system. Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 445–448 (1947). (Russian. English summary)

A mechanical system is characterized by the parameter ζ and the equation (1) $\ddot\zeta+2n\dot\zeta+\zeta=\mu$, where n is a positive constant and μ describes the position of a governing device whose movement, in turn, is determined by a three-way switch; the positions of the switch correspond to the three values of the quantity $\mathrm{sgn}\,\sigma+\mathrm{sgn}\,\dot\sigma$, where $\sigma=\zeta+\lambda\dot\zeta$ (λ=constant) represents the feedback of S into the control mechanism. The equation of the control mechanism is assumed to be (2) $\dot\mu=-\frac{1}{2}(\mathrm{sgn}\,\sigma+\mathrm{sgn}\,\dot\sigma)$. The author studies the stability of the system (1), (2) by means of a suitable change of variables and the application of Liapounoff's conditions. He claims that the condition $\lambda n>\frac{1}{2}$ is sufficient for stability but that there is not asymptotic stability.

However, the reviewer observes that since the second member of (2) is discontinuous Liapounoff's theorems are not immediately applicable. Moreover, it can be shown that the system has solutions, with initial points arbitrarily near the origin, which do not exist for all $t>0$, so that the question of stability loses its meaning and it is even doubtful if such differential equations may adequately represent any real mechanical system. The existence of relaxation motions is also apparently excluded by physical considerations.

J. L. Massera (Princeton, N. J.).

Volk, I. M. Generalizations of the method of small parameters in the theory of periodic motions of non-autonomous systems. Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 433–444 (1947). (Russian. English summary)

The author returns to the topic of a previous paper [same journal **10**, 559–574 (1946); these Rev. **8**, 330]. With the same notations he considers this time the case when the reduced system admits a solution with the period mT. He not only discusses the existence of the solution of the basic system but also its representation in series of powers of the parameter μ. *S. Lefschetz* (Princeton, N. J.).

Almeida Meneses, Pablo Rogerio. Subresonance for an equation of second order with forced and periodic vibrations, when the friction and the elastic force are not linear in the velocity. III. Revista Ci., Lima **49**, 201–238 (1947). (Spanish)

This is the third and apparently final installment in the publication of a thesis. The first two parts, which were devoted mainly to preliminaries, have been reviewed pre-

viously [same vol., 71–80, 87–166 (1947); these Rev. **8**, 583; **9**, 35]. In the present part the author employs an elementary method of successive approximations to obtain the conditions under which several particular differential equations of the general form $x''+\omega_n^2x+\mu f(x, x')=a \sin \omega t$ display the phenomenon of subresonance, i.e., the conditions under which the equations possess solutions of period $2\pi/\omega_n$ when ω is an integral multiple of ω_n. Neither the method nor the results call for any special comments.

L. A. MacColl (New York, N. Y.).

Gillis, Paul, et Van Hove, Léon. Un théorème d'unicité pour les systèmes d'équations différentielles du premier ordre. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 300–311 (1947).

Let $a_i(x, y, z)$ $(i=1, 2, 3)$ be continuous functions in some region D, and P a point of D at which some a_i is not zero. The question of uniqueness of the solution of the differential system (1) $dx/a_1=dy/a_2=dz/a_3$ passing through point P of D is considered. The author shows that if the symbolic form $\omega=a_1dydz+a_2dzdx+a_3dxdy$ has a nonvanishing integrating factor and if the equation $a_1f_x+a_2f_y+a_3f_z=0$ has a solution of class C^1 of which one derivative at least is not zero at point P, then (1) has a unique solution passing through point P. The results are generalized to n variables.

F. G. Dressel (Durham, N. C.).

Graff, A. To the theory of linear differential systems in one-dimensional domains. II. Rec. Math. [Mat. Sbornik] N.S. **21(63)**, 143–159 (1947). (Russian. English summary)

The present work is a sequel to part I [same Rec. **18(60)**, 305–328 (1946); these Rev. **8**, 74], in which a study is made of linear differential systems involving quasi-differential operators of order n. The author's methods are applicable to systems $L_n(y)=f(x)$, $A_i(y)=\gamma_i$ $(i=0, \cdots, n-1)$, where $L_n(y)=p_ny^{(n)}+\cdots+p_0y$, $A_i=A_i'+A_i''$, $A_i'(y)=\sum_1^{n-1}A_{i\nu}'(y)$; $A_{i\nu}'(y)=\sum_0^{n-1}(-1)^p\alpha_{i\nu p}y^{(p)}(a_\nu)$; $a_1, a_2, \cdots$ on (a, b); $A_i''(y)=\int_a^b\{\sum_0^{n-1}(-1)^p\alpha_{ip}(\lambda)y^{(p)}(\lambda)\}d\lambda$ (continuous $\alpha_{ip}(\lambda)$, defined for $a\leqq\lambda\leqq b$); these methods are also applicable to systems with operators $L_i(y_1, \cdots, y_n)=y_i^{(1)}+\sum_1^n a_{ij}y_j$ $(i=1, \cdots, n)$. Two systems $\sum_1^n a_{ik}x_k=0$ $(i=1, \cdots, n_1)$, $\sum_1^n b_{jk}\xi_k=0$ $(j=1, \cdots, n_2)$ are "bounded" if $\sum_{k=1}^n a_{ik}b_{jk}=0$. With the aid of this concept the following is proved. To construct a system of boundary conditions, "adjoint" to the given system $A_i(y)=B_i(y)$ $(i=0, \cdots, n-1)$, $A_i(y)=\sum_{k=0}^{n-1}(-1)^k\alpha_{ik}y^{(k)}(a)$, $B_i(y)=\sum_{k=0}^{n-1}(-1)^k\beta_{ik}y^{(k)}(b)$ (bilocal case), one needs merely to choose a corresponding set of solutions of the system $\sum_{k=0}^{n-1}(-1)^k\alpha_{ik}y_k^{(a)}=\sum_{k=0}^{n-1}(-1)^k\beta_{ik}y_k^{(b)}$ $(i=0, \cdots, n-1)$.

W. J. Trjitzinsky (Urbana, Ill.).

Saltykow, Nicolas. Intégration des équations aux différentielles totales linéaires à coefficients constants. C. R. Acad. Sci. Paris **225**, 520–521 (1947).

Restating the problem in matrix notation, the author considers the solution of $\partial y/\partial x^i=A_iy$ $(i=1, 2, \cdots, m)$ where A_i are constant matrices and y is the unknown vector. If the matrices A_i are commutative, the system is completely integrable and the solution is $y=e^{A_i(x^i-x_0^i)}y_0$, where y_0 is an arbitrary constant vector. If the matrices are not commutative a less general solution may or may not exist and the author considers the possible solution of the form (1) $y=\alpha\cdot e^{\theta_ix^i}$, where α is a constant vector and θ_i are m constants. This leads to (2) $(A_i-\theta_iI)\alpha=0$, which means that θ_i must be a root of the characteristic equation of A_i and α must be a characteristic vector common to all matrices. The paper gives the impression that a solution of the form (1) can always be found, which is not the case even if the system is completely integrable; for a necessary condition that (2) admit a common vector is that a characteristic root of A_iA_j be the product of the characteristic roots of A_i and of A_j, which in general is not the case.

M. S. Knebelman (Pullman, Wash.).

Szarski, Jacek. Sur un problème de caractère intégral relatif à l'équation: $\dfrac{\partial z}{\partial x}+Q(x, y)\dfrac{\partial z}{\partial y}=0$ définie dans le plan tout entier. Ann. Soc. Polon. Math. **19** (1946), 106–132 (1947).

The author shows there exists $Q(x, y)$ of class C^∞ in the entire plane such that the only C' solutions of the equation of the title are constants. A similar result is stated for a square.

D. G. Bourgin (Urbana, Ill.).

Vagner, V. On the concept of the indicatrix in the theory of partial differential equations. Doklady Akad. Nauk SSSR (N.S.) **57**, 219–222 (1947). (Russian)

The paper is concerned with the system of partial differential equations $\theta^u(\xi^a, \xi^p, \partial\xi^p/\partial\xi^a)=0$, $a=1, \cdots, m$; $p=m+1, \cdots, n$; $u=r+1, \cdots, m(n-m)$. The only restriction on θ^u is that they are analytic functions. Regarding ξ^a as functions of m parameters these equations become $F^u(\xi^a, \partial\xi^a/\partial t^\alpha)=0$, $\alpha=1, \cdots, n$; or if the solutions are looked for as implicit functions $\varphi^p(\xi^a)=\text{constant}$, the equations take the form $\phi^u(\xi^a, \partial\varphi^p/\partial\xi^a)=0$. The author claims certain "homogeneity" properties for the functions F^u and ϕ^u, which in general they do not possess. Regarding ξ^a as coordinates of a point in a space X_n, with each such point may be associated two local spaces of contravariant m-vectors and $(n-m)$-vectors from which the author constructs the space of all m-directions as an $m(n-m)$-dimensional surface in the local space of lines through ξ^a. This is the Grassmann surface at the point. A similar construction is carried through for the $(n-m)$-vectors. If the m-vectors are determined by the vectors $x_{(\alpha)}^a$ and the $(n-m)$-vectors by the covariant vectors $y_a^{(p)}$, the equations $F^u(\xi^a, x_{(\alpha)}^a)=0$; $\phi^u(\xi^a, y_a^{(p)})=0$ determine an r-dimensional surface in the Grassmann surface at each point of X_n, which is the r-dimensional indicatrix of the system of differential equations. If the equations of the contravariant and covariant indicatrices are solved for the vectors we get $x_{(\alpha)}^a=B_\alpha^a(\xi^a, \eta^i)$, $y_a^{(p)}=C_a^p(\xi^a, \eta^i)$, $i=1, \cdots, r$. [This involves assumptions on the independence of θ^u as functions of ξ_a^p which are not stated.] By introducing the affinor $g_{ia}^p=\partial_i(B_a^aC_a^p)$, the author shows that every integral surface of the given system can be obtained as the projection on X_n of an m-dimensional integral surface in X_{n+r} of the Pfaffian system $C_a^p(\xi^\lambda, \eta^i)d\xi^a=0$. The paper is concluded with Cartan's theorem for systems of partial differential equations in involution, which follows from the above discussion.

M. S. Knebelman.

Ghizzetti, A. Un'osservazione sul metodo di Ritz ed applicazione al calcolo della frequenza fondamentale di una membrana circolare con foro circolare eccentrico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 559–564 (1947).

The eigenvalues of the problem $E[v]+\lambda v=0$ in a given space D and $L[v]=0$ at the boundary FD are known; the eigenfunctions are v_n. Here $E[v]$ is a self-adjoint differential

expression of elliptic type:

$$E[v]=\sum_{h=1}^{r}\frac{\partial}{\partial x_h}\sum_{k=1}^{r}a_{hk}(X)\frac{\partial v}{\partial x_k}+c(X)v, \qquad c\leqq 0,$$

$$L[v]=\alpha(X)\partial v/\partial n'+\beta(X)v,$$

$$\alpha^2+\beta^2>0,\ \alpha\beta\leqq 0;$$

the direction n' is defined by

$$\alpha(X)\cos(n'x_k)=\sum_{h=1}^{r}a_{hk}(X)\cos(nx_h);$$

n designates the normal to FD. Solutions are required of $E[u]+\sigma\vartheta(X)u=0$ in D and $L[u]=0$ on FD. It is proved that the least eigenvalue σ_0 is given by

$$\sigma_0=\lim_{N\to\infty}I\left[\sum_0^N\gamma_n^Nv_n\right]\bigg/H\left[\sum_0^N\gamma_n^Nv_n\right],$$

where

$$I[u]=\int_D\left(\sum_{h=1}^{r}\sum_{k=1}^{r}a_{hk}\frac{\partial u}{\partial x_h}\frac{\partial u}{\partial x_k}-cu^2\right)dX$$

$$-\int_{FD}a\frac{\alpha\beta}{\alpha^2+\beta^2}[u^2+(\partial u/\partial n)^2]d\sigma,$$

$$H[u]=\int_D\vartheta u^2dX,$$

and $\gamma_0^N, \gamma_1^N, \cdots, \gamma_N^N$ are the values of $\gamma_0, \gamma_1, \cdots, \gamma_N$ that minimize $I[\sum_0^N\gamma_nv_n]/H[\sum_0^N\gamma_nv_n]$. The result is applied to find the lowest characteristic period of a circular membrane with an eccentric circular hole. *H. Bremekamp* (Delft).

Bergman, S., and Schiffer, M. A representation of Green's and Neumann's functions in the theory of partial differential equations of second order. Duke Math. J. 14, 609–638 (1947).

L'auteurs étudient d'abord dans un domaine plan ω, l'équation $\Delta u=P(x, y)u$, où P est une fonction finie continue positive du point $Z(x, y)$, et si

$$D_E(u, v)=\iint_\omega(\text{grad } u\cdot\text{grad } v+Puv)dxdy,$$

ils considèrent les solutions u pour lesquelles $\delta_u=D(u, u)$ est fini [même si Δ est pris en un sens généralisé; cela semble impliquer des conditions sur P qui est peut-être ainsi donné dans un ouvert contenant ω et sa frontière; il me paraît d'autre part probable que l'hypothèse $P>0$ puisse être élargie pour la suite, en $P\geqq 0$ mais non partout nulle dans ω]. Les auteurs rappellent que Bergman à montré [Duke Math. J. 14, 349–366 (1947); ces Rev. 9, 181] la représentation de toute $u(Z)$ par une série $\sum A_\nu\varphi_\nu(Z)$ où les φ_ν forment un système orthonormal complet fixé de fonctions u et que $\sum\varphi_\nu(Z)\varphi_\nu(W)$ est indépendant du choix de ce système. C'est le noyau $K(Z, W)$. Les auteurs en donne deux caractères. (a) Parmi les u égales à 1 en W, il y en a une seule réalisant le minimum δ de δ_u et c'est $\delta K(Z, W)$ avec $\delta=1/K(W, W)$. (b) Si $L(Z, W)$ symétrique est pour chaque variable un u, l'identité $u(Z)=D_W[L(Z, W), u(W)]$ pour toute u entraîne $L=K$. Avec une frontière assez régulière, K est liée aux fonctions G de Green et N de Neumann (avec pôle en log ZW et nullité à la frontière de G ou de dN/dn), selon $2\pi K=N-G$. Puis $D_W[G(Z, W), u(W)]=0$ et $D_W[N(Z, W), u(W)]=2\pi u(Z)$, ce qui est interprété géométriquement en espace fonctionnel. On examine ensuite le cas harmonique analogue mais assez différent où interviennent les fonctions conjuguées et la mesure harmonique de chacune des composantes connexes de la frontière (en nombre fini). On étudie encore la manière dont le noyau dépend de ω en retrouvant les formules classiques de Hadamard pour G et N. Des extensions de la théorie sont indiquées pour des types d'équations plus généraux, pour un nombre quelconque de variables, enfin pour des équations de degré plus élevé comme $\Delta\Delta u=0$. *M. Brelot.*

Khalilov, Z. I. Sur les problèmes aux limites pour l'équation elliptique. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 11, 345–362 (1947). (Russian. French summary)

The equation studied is

$$(1)\qquad L(u)\equiv u_{xx}+u_{yy}+Au_x+Bu_y+Cu=0,$$

where A, B, C are functions of $(x, y; \lambda)$, entire in x, y in a simple domain S, bounded by a contour L with defined curvature at each point; A, B, C are analytic in λ for $\lambda_1\leqq\lambda\leqq\lambda_2$. The boundary condition is

$$(2)\qquad R(u)\equiv\sum a_{p,q}(s, \lambda)[D_{p,q}u]_s=b(s, \lambda)$$

(sum over $0\leqq p+q\leqq N$; $D_{p,q}=\partial^{p+q}/\partial x^p\partial y^q$; the $a_{p,q}$, b are analytic in λ on (λ_1, λ_2) and are of a Hölder class in s (on L)). Problem (A) is: given λ on (λ_1, λ_2), find a real regular u satisfying (1) (in S), (2) (on L). Problem (A^0) is (A) with $b\equiv 0$. Use is made of Vekua's theory of singular integral equations with Cauchy kernels; on the basis of this theory a number called ind L is introduced. Problem (A) is solved explicitly. Some of the results are as follows. The rank of every characteristic value of (A^0) is finite. Problem (A) has a solution, for a given λ on (λ_1, λ_2), if and only if b is orthogonal (on L) to the characteristic functions of the equation adjoint to $L(\varphi)=0$. If ind for (A^0) is zero, then (A) has a solution if (A^0) has $u=0$ as the only solution. If ind (A)$\geqq 0$, then the spectrum (A^0) is either coincident with the interval (λ_1, λ_2) or is discrete. *W. J. Trjitzinsky* (Urbana, Ill.).

Vekua, I. N. On a generalization of the Poisson integral for a half-plane. Doklady Akad. Nauk SSSR (N.S.) 56, 229–231 (1947). (Russian)

Considérons l'équation: (1) $y^{2k}u_{xx}+u_{yy}=0$ ($k>-1$ étant une constante réelle) et soit une fonction $f(x)$, définie, bornée et continue par morceaux pour $-\infty<x<\infty$. L'auteur se propose de former la solution $u(x, y)$ de (1), définie dans le domaine $y>1$ du plan Oxy, telle que $u(x, 0)=f(x)$ pour chaque valeur de x où $f(x)$ est continue. En posant:

$$\Lambda=\Gamma(\tfrac{1}{2})\Gamma\left(\frac{1}{2k+2}\right)\bigg/\Gamma\left(\frac{k+2}{2k+2}\right)$$

l'auteur prouve que l'expression

$$(2)\quad u(x, y)=\frac{1}{\Lambda y^{k+1}}\int_{-\infty}^{\infty}f(\xi)[(x-\xi)^2+y^{2k+2}/(2k+2)]^{-(k+2)/(2k+2)}d\xi,$$

répond à la question; l'unicité de la solution trouvée est garantie par les théorèmes généraux concernant les équations du type elliptique. A noter que pour $y>0$, on a: $|u(x, y)|\leqq\max|f(x)|$.

Si on particularise les valeurs de k on retrouve des résultats connus [$k=0$, par exemple, correspond à l'équation de Laplace: $\Delta u=0$, auquel cas (2) se réduit à la formule de Poisson pour le demi-plan; pour $k=\frac{1}{2}$, on retrouve l'équation de Tricomi-Darboux, étudiée dans un récent mémoire de

F. Frankl [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 135–166 (1946); ces Rev. **8**, 77].
J. Kravtchenko (Grenoble).

✱Carslaw, H. S., and Jaeger, J. C. Conduction of Heat in Solids. Oxford, at the Clarendon Press, 1947. viii+386 pp. $8.00.

This is a new book based on Carslaw's "Introduction to the Mathematical Theory of the Conduction of Heat in Solids" [Macmillan, London, 1921]. Roughly, one third of the contents of the book consists of new material which brings the theory and applications up to date. The method of contour integration has been replaced by the use of the Laplace transformation. Practical applications are mentioned throughout the book and, in several cases, numerical and graphical results are given. The chapter headings include: general theory; linear flow of heat(three chapters); flow of heat in a rectangle, in a rectangular parallelopiped, in regions bounded by the surfaces of the cylindrical coordinate system, in a sphere and cone; the use of sources and sinks; the use of the Laplace transformation including its applications to problems on the cylinder and sphere and to the determination of Green's functions (four chapters); steady temperatures. The appendices contain useful tables of the error function and related functions, a summary of properties of Bessel functions, tables of roots of certain transcendental equations including the equations $x \tan x = C$ and $x \cot x = C$, a short table of Laplace transforms and a table of thermal properties of some common substances. The book is clearly and concisely written. It should prove to be interesting and useful to engineers and physicists as well as to mathematicians.
R. V. Churchill.

Green, George. Solutions of problems relating to media in contact by the method of wave-trains. Philos. Mag. (7) **38**, 97–115 (1947).

Media (1) and (2) have boundary surfaces A, B and B, C, respectively. These are planar, cylindrical or spherical. The heat conduction problem is considered. Source effects are studied. The author takes the periodic source solutions as fundamental and obtains the instantaneous and continuous solutions by integration from these. The method is discussed in the reviewer's comments on one of the author's earlier papers [these Rev. **6**, 156].
D. G. Bourgin.

Krasnuškin, P. E. The method of normal waves with an application to plane-stratified media. Doklady Akad. Nauk SSSR (N.S.) **56**, 687–690 (1947). (Russian)

In this paper there are considered wave phenomena which arise under the action of given currents of density J in an unbounded isotropic nonmagnetic medium, whose dielectric constant ϵ is a function of one coordinate z only. If cylindrical coordinates (r, θ, z) are used, and if it is assumed that the currents are directed along the z-axis and do not depend on the angle θ, then the wave phenomena are described by the scalar differential equation

$$(1)\qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)+\epsilon k^2\frac{\partial}{\partial z}\left(\frac{1}{\epsilon k^2}\frac{\partial A}{\partial z}\right)+\epsilon k^2 A=-(4\pi/c)J(r,z),$$

where A is the component of the potential vector A_z, and k is the wave number. Then A is represented in the form of the spectrum

$$(2)\qquad A=\sum_j \varphi_j(r)Z_j(z)+\int \varphi(r,X)Z(z,X)dX,$$

where $Z_j(z)$ and $Z(z, X)$ are characteristic functions of the discrete and the continuous parts of the spectrum, and are determined by the equation

$$(3)\qquad \epsilon k^2\frac{d}{dz}\left(\frac{1}{\epsilon k^2}\frac{dZ}{dz}\right)+\epsilon k^2 Z+XZ=0$$

under the conditions that the function Z is single valued and continuous. The functions Z are assumed to have been normalized in the usual way. Substituting (2) into (1) and taking into account (3) and the normalization, there results

$$(4)\qquad \frac{d}{dr}\left(r\frac{d\varphi}{dr}\right)-Xr\varphi=-(4\pi/c)rJ(r),$$

where $J(r)=\int J(r,z)Z(z)dz$. The solution $\varphi(r)$ of (4) is

$$\varphi(r)=(-i\pi^2/c)\int_0^\infty K(r,\rho)\rho J(\rho)d\rho,$$

where $K(r, \rho)$ is Green's function given by

$$J_0((-X\rho)^{\frac{1}{2}})H_0^{(2)}((-rX)^{\frac{1}{2}}) \text{ if } r>\rho;$$
$$H_0^{(2)}(z(-X\rho)^{\frac{1}{2}})J_0((-Xr)^{\frac{1}{2}}) \text{ if } r<\rho;$$

J_0 and H_0 are Bessel's and Hankel's functions of order zero.

It is pointed out that the cases of discrete spectra, namely the cases when the medium has the properties of a wave conductor, are of special interest. The simplest case is that of a homogeneous layer with $\epsilon=1$, bounded by perfectly conducting planes. It is shown in this paper that any nonhomogeneous medium, with ϵ a continuous function of z only, possesses the property of a wave conductor whenever the spectrum is discrete. The case of two parallel wave conducting channels is also considered. In this case it is shown that the energy can jump back and forth from one channel to the other.
H. P. Thielman (Ames, Iowa).

Bureau, Florent. Le problème de Cauchy et la théorie de la propagation des ondes lumineuses dans les milieux cristallins homogènes et uniaxes. C. R. Acad. Sci. Paris **225**, 402–403 (1947).

This note contains a solution of the Cauchy problem for the fourth order hyperbolic differential equation $D_1D_2u=f(x_1, x_2, x_3, t)$, where D_1, D_2 are the second order operators

$$D_1=\frac{\partial^2}{\partial t^2}-a^2\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right),$$

$$D_2=\frac{\partial^2}{\partial t^2}-c^2\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)-a^2\frac{\partial^2}{\partial x_3^2},\qquad 0<c<1.$$

In terms of the Cauchy data on a space-like surface S, the expression for u takes the form of the "logarithmic part" of an integral extended over that portion of S that is cut out by the characteristic cone belonging to D_1. The method of derivation appears to be the same as that used previously by the author [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 185–199 (1947); these Rev. **9**, 95] for the analogous case where D_1 and D_2 are second order operators differing only in the coefficients of $\partial^2/\partial t^2$.
F. John (New York, N. Y.).

Difference Equations, Special Functional Equations

Kestelman, H. On the functional equation $f(x+y)=f(x)+f(y)$. Fund. Math. **34**, 144–147 (1947).

The author gives a simple proof of Ostrowski's result [Jber. Deutsch. Math. Verein. **38**, 54–62 (1929)] that a real

solution of $f(x+y)=f(x)+f(y)$ is linear if bounded above on a set E of positive measure. The proof uses Steinhaus's theorem [Fund. Math. 1, 93–104 (1920)] that all sufficiently small positive numbers are differences of points of E; a short and elementary proof (not involving density) is given for this theorem in Euclidean n-space. It is pointed out that Steinhaus's theorem also holds for sets E (possibly of measure zero) such that there is a sphere S with $S-ES$ of first category. Let $U(x)$ be an additive operation with domain and range in normed linear spaces. If $U(x)$ is bounded on a set E with the property of Steinhaus's theorem (i.e., the "set of distances" of E contains a sphere), then $U(x)$ is homogeneous. *R. P. Boas, Jr.*

Shah, S. M. On real continuous solutions of algebraic difference equations. Bull. Amer. Math. Soc. 53, 548–558 (1947).

The author studies at infinity real continuous solutions of (1) $P(y(x+m), \cdots, y(x), x)=0$, where P is a polynomial (with real coefficients) in the arguments. He improves a result of O. E. Lancaster [Bull. Amer. Math. Soc. 46, 169–177 (1940); these Rev. 1, 181] proving that any real continuous solution of (1), with $m=1$, satisfies

$$\liminf \log\log |y(x)|x^{-1}<\infty$$

(as $x\to\infty$); this result, in a sense, is the best possible. Similar results are proved for any m. The developments presented by the author constitute a continuation of a line of investigation carried on by Borel and Vijayaraghavan, among others. *W. J. Trjitzinsky* (Urbana, Ill.).

Cooper, R. A class of recurrence formulae. J. London Math. Soc. 22, 31–40 (1947).

The recurrence formula is $\varphi(n)c_n=\sum_{m=1}^{n-1}c_mc_{n-m}\psi(m)$, $n\geqq 2$, with $\psi(n)$ decreasing to zero and for some $\delta>0$, $r>0$ and $a\geqq 1$, $n^\delta=o(\varphi(n))$, $\varphi(n)=o(\exp r(\log n)^a)$. The radius of convergence of the series $\sum_{n=1}^{\infty}c_nx^n$ is shown to be finite or infinite depending on certain order conditions on $\psi(n)$. This work is a generalization of that of E. M. Wright [same J. 20, 68–73 (1945); these Rev. 7, 431]. *P. Civin.*

Integral Equations

Hebroni, P. On relations existing between two kernels of the form $(a, b)+b$ and $(b, a)+b$. Bull. Amer. Math. Soc. 53, 753–756 (1947).

This paper is devoted to a consideration of the formal relations between the kernels of integral equations, their resolvents and their Fredholm determinants. The main result, stated in operator notation, is that if A and B are linear operators (presumably with continuous kernels, though this is not stated), then $AB+B$ and $BA+B$ have the same Fredholm determinant.

The result could be obtained more directly by observing that $(AB+B)^n$ and $(BA+B)^n$ have equal traces for every positive integer n, and using the expression for the Fredholm determinants in terms of these traces. *F. Smithies.*

Parodi, Maurice. Sur un type d'équations intégrales résolubles par le calcul symbolique. C. R. Acad. Sci. Paris 226, 43–45 (1948).

The author solves the integral equation

$$f(t)+\lambda\int_0^\infty K(t, x)f(x)dx=g(t)$$

(K and g given) formally by means of the Laplace transformation when the Laplace transform of $K(t, x)$ is of the form $\varphi(p)e^{-x\psi(p)}$, where φ and ψ are independent of x and $\psi(\psi(s))=s$. He applies his method to the Hankel transformation and to Fourier's sine transformation without finding essentially new results. Possible generalisations are indicated. *A. Erdélyi* (Pasadena, Calif.).

Chandrasekhar, S. The transfer of radiation in stellar atmospheres. Bull. Amer. Math. Soc. 53, 641–711 (1947).

This paper contains the twentieth Gibbs lecture to the American Mathematical Society. The author begins by showing how the equation of radiative transfer can be obtained for scattering atmospheres of various types. When the polarization of the scattered radiation is taken into account, the equation becomes a set of integro-differential equations, which can be written as a single vector equation. After giving an account of a method based on the Gauss quadrature formula for obtaining numerical approximations to the solution of the equation of transfer, he shows that its exact solution can be obtained by solving appropriate equations of the form

$$H(\mu)=1+\mu H(\mu)\int_0^1\frac{H(\mu')\Psi(\mu')}{\mu+\mu'}d\mu', \tag{1}$$

where $\Psi(\mu)$ is an even polynomial in μ such that $\int_0^1\Psi(\mu)d\mu\leqq\frac{1}{2}$. Various properties of the functions $H(\mu)$ are established, and in an addendum to the paper he states that M. M. Crum has obtained a solution of (1) in the form

$$\log H(\mu)=(2\pi i)^{-1}\int_{-i\infty}^{i\infty}\log T(z)\frac{\mu dz}{z^2-\mu^2},$$

where $T(z)=1-2z^2\int_0^1(z^2-\mu^2)^{-1}\Psi(\mu)d\mu$.

The paper concludes with brief discussions of the problem for the cases of a spherical atmosphere and a differentially moving atmosphere. In the latter case the problem leads to a hyperbolic partial differential equation with an unusual type of boundary condition, and the author shows how it can be solved by the introduction of appropriately defined Green's functions.

The addenda contain a direct derivation of a relationship between the law of darkening and the law of diffuse reflection, and a brief discussion of the case of an atmosphere of finite optical thickness, when equation (1) is replaced by a pair of equations for two unknown functions.

F. Smithies (Cambridge, England).

Marshak, R. E. The Milne problem for a large plane slab with constant source and anisotropic scattering. Physical Rev. (2) 72, 47–50 (1947).

This paper deals with the solution of the equation

$$\mu\frac{\partial\psi(z, \mu)}{\partial z}+\psi(z, \mu)=\frac{1}{2\sigma}\left\{\int_{-1}^1\psi(z, \mu')d\mu'+3f_1\mu\int_{-1}^1\psi(z, \mu')\mu'd\mu'\right\}+\tfrac{1}{2}q_0,$$

which satisfies the boundary conditions $\psi(0, \mu)=0$ for $\mu>0$ and

$$\frac{\partial}{\partial z}\int_{-1}^1\psi(z, \mu)d\mu=0, \qquad z=d,$$

where σ, q_0, f_1, and d are certain assigned constants, by an adaptation of the method of Hopf and Wiener [cf. E. Hopf, "Mathematical Problems of Radiative Equilibrium," Cam-

bridge University Press, 1934]. The method of obtaining explicit solutions for the case $d\to\infty$ is outlined. [For the exact solution of a closely related problem see the paper reviewed above, particularly the solution under C in § 22 and the representation of the H-functions as complex integrals in § 35a.] *S. Chandrasekhar* (Williams Bay, Wis.).

Marshak, Robert E. Theory of the slowing down of neutrons by elastic collision with atomic nuclei. Rev. Modern Physics **19**, 185–238 (1947).

This paper is principally concerned with the solution of the following equations in the theory of neutron diffusion:

$$(*)\quad \frac{l(u)}{v}\frac{\partial\psi}{\partial t}+l(u)\boldsymbol{\Omega}\cdot\operatorname{grad}\psi+\psi(\mathbf{r},\boldsymbol{\Omega},u,t)$$
$$=\int_0^\infty du'\int d\boldsymbol{\Omega}'\psi(\mathbf{r},\boldsymbol{\Omega}',u',t)f(\mu_0,u-u')h(u')+Q(\mathbf{r})\delta(u)\delta(t)$$

and

$$(**)\quad \frac{l(u)}{v}\frac{\partial\Psi_0}{\partial t}+\Psi_0(u,t)=\int_0^\infty du'\Psi_0(u',t)h(u')f_0(u-u')+Q\delta(u)\delta(t).$$

In $(*)$, $\psi(\mathbf{r},\boldsymbol{\Omega},u,t)d\mathbf{r}d\boldsymbol{\Omega}du$ denotes the total number of collisions per unit time between $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$, etc., $\boldsymbol{\Omega}$ a unit vector in the direction of the velocity of magnitude v, $u=\log(E_0/E)$, where E_0 is some initial energy, $l(u)$ the mean free path, $h(u)$ the probability of capture,

$$f(\mu_0,u)=\frac{(M+1)^2}{8\pi M}e^{-u}\delta\{\mu_0-[\tfrac{1}{2}(M+1)e^{-u/2}-\tfrac{1}{2}(M-1)e^{u/2}]\},$$

$\delta(x)$ the Dirac δ-function, $\mu_0=\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}'$, $Q(\mathbf{r})$ is a function of $\mathbf{r}$ alone and M is the mass of the colliding particle in the units of the neutron mass. The second equation governs the spatial distribution of neutrons and is obtained by integrating $(*)$ over all space:

$$\Psi_0(u,t)=\int d\mathbf{r}\int d\boldsymbol{\Omega}\psi(\mathbf{r},\boldsymbol{\Omega},u,t),\quad Q=\int d\mathbf{r}Q(\mathbf{r}),$$

and $f_0(u)=(M+1)^2(4M)^{-1}e^{-u}$ for

$$u\leqq q_M=\log\{(M+1)/(M-1)\}^2$$

and zero otherwise.

Various cases of the foregoing equations are considered. The equations are generally solved by the method of Laplace transform. Thus, in the stationary case of "no capture," the equation to be solved is

$$\Psi_0(u)=\int_0^\infty du'\Psi_0(u')f_0(u-u')+\delta(u).$$

Denoting by $\Phi_0(\eta)$ and $G_0(\eta)$ the Laplace transforms of Ψ_0 and f_0, we have by the "convolution theorem" $\Phi_0(\eta)=\Phi_0(\eta)G_0(\eta)+1$, where

$$G_0(\eta)=\frac{\alpha}{\eta+1}\{1-e^{-q_M(\eta+1)}\},\quad \alpha=\frac{(M+1)^2}{4M}.$$

Solving for $\Phi_0(\eta)$ and inverting the Laplace transform, we have

$$\Psi_0(u)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{\eta u}d\eta}{1-[\alpha/(\eta+1)][1-e^{-q_M(\eta+1)}]},$$

where the integration is taken over a line to the right of all the poles of the integrand. The poles of the integrand are at $(\eta+1)/\alpha=1-e^{-q_M(\eta+1)}$. Similar methods apply in the more general cases. *S. Chandrasekhar* (Williams Bay, Wis.).

Placzek, G. Notes on diffusion of neutrons without change in energy. Notes on a series of seminars, recorded and extended by G. M. Volkoff. National Research Council of Canada. Division of Atomic Energy. Document no. **MT-4**, v+32+10 pp. (1943).

It is well known that the integrodifferential equation

$$\text{(i)}\quad \operatorname{div}[\boldsymbol{\Omega}\psi(\mathbf{r},\boldsymbol{\Omega})]=-\psi(\mathbf{r},\boldsymbol{\Omega})+(1-\alpha)\psi_0(\mathbf{r})+q(\mathbf{r},\boldsymbol{\Omega}),$$

where $\boldsymbol{\Omega}$ denotes a unit vector, α is a constant and

$$\psi_0(\mathbf{r})=\int\psi(\mathbf{r},\boldsymbol{\Omega})d\boldsymbol{\Omega},$$

can be reduced to the integral equation

$$\text{(ii)}\quad \psi_0(\mathbf{r})=\frac{1-\alpha}{4\pi}\int\left\{\psi_0(\mathbf{r}')+\frac{4\pi}{1-\alpha}q\left(\mathbf{r}',\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\right)\right\}\frac{e^{-|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|^2}d\mathbf{r}'.$$

In equation (ii) the integration is extended over the entire medium.

When the medium is of infinite extent, equation (ii) can be solved by the method of Fourier transforms. Thus, if $\phi(\mathbf{k},\boldsymbol{\Omega})$, $\phi_0(\mathbf{k})$ and $\lambda(\mathbf{k},\boldsymbol{\Omega})$ are the three-dimensional Fourier transforms of $\psi(\mathbf{r},\boldsymbol{\Omega})$, $\psi_0(\mathbf{r})$ and $q(\mathbf{r},\boldsymbol{\Omega})$, respectively, we readily find from equation (i) that

$$\phi(\mathbf{k},\boldsymbol{\Omega})=\frac{1}{1-\mathbf{k}\cdot\boldsymbol{\Omega}}\left\{\frac{1-\alpha}{4\pi}\phi_0(\mathbf{k})+\lambda(\mathbf{k},\boldsymbol{\Omega})\right\}$$

and

$$\phi_0(\mathbf{k})=(1-\alpha)k^{-1}\tan^{-1}k\phi_0(\mathbf{k})+\int\frac{\lambda(\mathbf{k},\boldsymbol{\Omega})}{1-i\mathbf{k}\cdot\boldsymbol{\Omega}}d\boldsymbol{\Omega}.$$

Inverting this last equation we have the following solution of the integral equation (ii) in the case where the integral on the right-hand side is extended over the entire r-space:

$$\psi_0(\mathbf{r})=\frac{1}{(2\pi)^3}\int d\mathbf{k}\frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{1-(1-\alpha)k^{-1}\tan^{-1}k}\int\frac{\lambda(\mathbf{k},\boldsymbol{\Omega})}{1-i\mathbf{k}\cdot\boldsymbol{\Omega}}d\boldsymbol{\Omega}.$$

For the case of a unit isotropic point source $(q(\mathbf{r})=\delta(r)/4\pi r^2)$ the solution reduces to

$$\psi_0(r)=\frac{1}{2\pi^2 r}\int_0^\infty\frac{\tan^{-1}k}{1-(1-\alpha)k^{-1}\tan^{-1}k}\sin kr\,dk.$$

This solution is discussed in some detail in an appendix.

The "notes" contain in addition the discussion of a variety of related matters such as the relationship of the exact treatment based on the transport equation (or the equation of transfer) and the approximate treatment based on the ordinary diffusion equation, etc. A number of useful formulae in the theory of Fourier transforms are collected in an appendix. *S. Chandrasekhar* (Williams Bay, Wis.).

Kourganoff, Vladimir. Sur la constance du flux intégré dans les atmosphères stellaires et la résolution de l'équation de transfert. C. R. Acad. Sci. Paris **225**, 491–493 (1947).

The following variational principle for solving the equation of transfer

$$\mu dI(\tau,\mu)/d\tau=I(\tau,\mu)-\tfrac{1}{2}\int_{-1}^{1}I(\tau,\mu')d\mu'$$

is suggested. A solution of the form

$$B(\tau)=\tfrac{1}{2}\int_{-1}^{1} I(\tau,\mu)d\mu=\sum_i b_i\tau^i/i!+\sum_n a_n e^{-\beta_n\tau}$$

is assumed and the constants b_i, a_n and β_n are determined by the condition that

$$\int_0^\infty [F(\tau)/F(0)-1]^2 d\tau=\text{minimum},$$

where

$$\tfrac{1}{4}F(\tau)=\int_{-1}^{1} I(\tau,\mu)\mu d\mu$$
$$=\int_\tau^\infty B(t)E_2(t-\tau)dt-\int_0^\tau B(t)E_2(\tau-t)dt.$$

Here $E_2(x)=\int_1^\infty t^{-2}e^{-xt}dt$ is the second exponential integral.
S. Chandrasekhar (Williams Bay, Wis.).

Functional Analysis, Ergodic Theory

Nakano, Hidegorô. Stetige lineare Funktionale auf dem teilweisegeordneten Modul. J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. 4, 201–382 (1942).

A "teilweisegeordneter Modul" is a partially ordered real vector space $\mathfrak{M}$ which is σ-conditionally complete. To each vector a corresponds an absolute vector $|a|$ and a projector $[a]$ which is a linear operator in $\mathfrak{M}$. The projectors form a Boolean ring and can therefore be identified with the Borel ring of bicompact-and-open subsets of a suitable basic space M. For certain linear subspaces $\mathfrak{N}\subset\mathfrak{M}$, the basic spaces of the $\mathfrak{N}$ can be identified with the open subsets $N\subset M$, the basic spaces of the quotients $\mathfrak{M}/\mathfrak{N}$ with the subsets $M-N$. A real valued linear functional $L(a)$ is continuous if $L(a_n)\to L(a)$ whenever $a_n\to a$ in the order topology of $\mathfrak{M}$. To each continuous linear L corresponds the null vector space $\mathfrak{N}(L)$ of vectors a for which $L[a]=0$ (that is, $L([a]x)=0$ for all x in $\mathfrak{M}$), and the associated $N(L)\subset M$. By definition, $L\geqq 0$ means $L(a)\geqq 0$ for every $a\geqq 0$; L_1 is absolutely continuous with respect to L means $L_1[a]=0$ whenever $L[a]=0$; L_1 is singular with respect to L means that for every a there is a projector $[p]$ such that $L_1(a)=L_1([p]a)$ but $L[p]=0$.

The author shows that the continuous linear functionals form a partially ordered vector space $\mathfrak{M}^*$ which is conditionally complete. For L_1 to be absolutely continuous with respect to L is equivalent to $[L_1]\leqq[L]$ in $\mathfrak{M}^*$, and also to $N(L)\subset N(L_1)$ in M. For L_1 to be singular with respect to L is equivalent to $|L_1|\cap|L|=0$ in $\mathfrak{M}^*$, and also to $N(L_1)+N(L)=M$. The decomposition $L_1=([L]L_1)+(L_1-[L]L_1)=A+S$ (say) gives the unique decomposition, à la Lebesgue, of L_1 into a sum such that A is absolutely continuous and S is singular with respect to L.

If L_1 is absolutely continuous with respect to L, then every projector $[a]$ has a spectral resolution $[a_\lambda]$, $-\infty<\lambda<+\infty$, such that, à la Hilbert space spectral theory,

$$L_1(x)=\int_{-\infty}^{+\infty}\lambda dL([a_\lambda]x)$$

for all x with $[a]x=x$.

For every fixed a, L defines in M a set function L^a: $L^a[p]=L([p]a)$. This L^a is defined over the Borel ring of bicompact-and-open subsets of M, and is totally additive. If P is a point in $M-N(L)$, the derivative of L_1^a with respect to L^a at P can be defined for all a with P in $[a]$. This numerical derivative $\varphi(P)$ is independent of a. The author develops a suitable integration theory (Riemann) for M such that the absolutely continuous part of L_1 with respect to L can be expressed as $\int\varphi(P)L(dP)$.

The above theory is worked out explicitly for the case: S an abstract space with a given Borel ring of measurable subsets and $\mathfrak{M}$ the set of all finite valued measurable functions $f(x)$, with the ordering: $f\geqq g$ means that $f(x)\geqq g(x)$ for all x in S.

Now assume that $\mathfrak{M}$ is conditionally complete. The author calls L completely continuous if it has the further property: whenever $\wedge(a_\alpha;\alpha\epsilon I)=0$ for any set of a_α, then $\inf|L(\wedge(a_\beta;\beta\epsilon J))|=0$, where J runs over all finite subsets of I. He calls $\mathfrak{M}$ semi-regular if for every $a\neq 0$, $L(a)\neq 0$ for some completely continuous L. Then the completely continuous L's form a semi-regular conditionally complete partially ordered vector space $\overline{\mathfrak{M}}\subset\mathfrak{M}^*$. If $\mathfrak{M}$ is semi-regular, then $\mathfrak{M}$ can be considered as a subspace of $\overline{\overline{\mathfrak{M}}}$, and it is called order-reflexive if $\mathfrak{M}=\overline{\overline{\mathfrak{M}}}$. The $\overline{\mathfrak{M}}$ is always order-reflexive. A necessary and sufficient condition that $\mathfrak{M}$ be semi-regular and order-reflexive is this: any set of vectors a_α, $\alpha\epsilon I$, $a_\alpha\geqq 0$, must be bounded in $\mathfrak{M}$, if for every completely continuous $L\geqq 0$, the numbers $L(\vee(a_\beta;\beta\epsilon J))$ are bounded when J runs over all finite subsets of I.

The author then considers those $\mathfrak{M}$ which are complete with respect to a norm $\|a\|$ (Banach spaces), such that $|a|\leqq|b|$ implies that $\|a\|\leqq\|b\|$. A sufficient condition is formulated under which $\overline{\mathfrak{M}}$ coincides with the usual Banach conjugate space, the condition involving continuity of $\|a\|$ with respect to $|a|$. A sufficient condition is formulated under which $\mathfrak{M}$ is order-reflexive, the condition involving a uniformity property of $\|a+b\|$ when $a\cap b=0$. The uniformity property of $\|a+b\|$ is then expressed in terms of the indicatrices of $\mathfrak{M}$: an indicatrix is defined for every pair of vectors a, b with $\|a\|=\|b\|=1$, $a\cap b=0$, and consists of all number-pairs (λ,μ) satisfying $\|\lambda a+\mu b\|=1$.

Special cases of such partially ordered Banach spaces are the $\mathfrak{L}_p$ $(p\geqq 1)$, for which $a\cap b=0$ implies $\|a+b\|^p=\|a\|^p+\|b\|^p$, and the $\mathfrak{L}_\infty$ for which $a\cap b=0$ implies $\|a+b\|=\max(\|a\|,\|b\|)$. If a partially ordered Banach space $\mathfrak{M}$ has at least three nonzero elements a, b, c such that $a\cap b=b\cap c=c\cap a=0$, then $\mathfrak{M}$ is an $\mathfrak{L}_p$ $(1\leqq p\leqq+\infty)$, if and only if $\mathfrak{M}$ has only one indicatrix. If, in addition, $\mathfrak{M}$ is separable, then it can be identified with L_p: space of $f(x)$, measurable on $(0, 1)$, $\int_0^1|f(x)|^p dx$ finite; or with l_p: space of sequences (a_n), $\sum_{n=1}^\infty|a_n|^p$ finite; or with the direct sum of L_p and l_p. For a prior characterization of these spaces the author refers to F. Bohnenblust [Duke Math. J. 6, 627–640 (1940); these Rev. 2, 102]. *I. Halperin* (Kingston, Ont.).

de Sz. Nagy, Béla. On uniformly bounded linear transformations in Hilbert space. Acta Univ. Szeged. Sect. Sci. Math. 11, 152–157 (1947).

The author considers linear transformations T in Hilbert space such that the powers of T are uniformly bounded, $|T^n|\leqq k$, $n=0,\pm 1,\pm 2,\cdots$. For these he establishes the existence of a self-adjoint transformation Q satisfying $k^{-1}I\leqq Q\leqq kI$ and such that QTQ^{-1} is a unitary transformation. The analysis of uniformly bounded transformations in reflexive spaces was given by the reviewer [Trans. Amer. Math. Soc. 49, 18–40 (1941); these Rev. 2, 224]. The present work shows that for Hilbert space no new class of

transformations is thereby analysed since the structure of unitary transformations is fully known.

The proof is based on the properties of a generalized limit due to Mazur and Banach and defined for all bounded complex functions $\xi(s)$, where s is a positive real variable or a positive integer. It is shown that the quantity $\lim (T^s f, T^s g) = \langle f, g\rangle$ is a Hermitian bilinear form and hence of the form (Af, g), where A is self-adjoint, $k^{-2}I \leqq A \leqq k^2 I$. Note that $(T^n f, T^n g) = \xi(n)$ is bounded by hypothesis. To establish the theorem it suffices to take $Q = A^{\frac{1}{2}}$.

Mazur's limit requires the well-ordering theorem. However, for a separable Hilbert space, one may avoid it. All the preceding results apply essentially to uniformly bounded continuous groups T, with $-\infty < s < \infty$. *E. R. Lorch.*

Rellich, F. Der Eindeutigkeitssatz für die Lösungen der quantenmechanischen Vertauschungsrelationen. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1946**, 107–115 (1946).

L'auteur prouve, en la précisant, une hypothèse d'unicité due à Born, Heisenberg et Jordan: (I) P et Q sont deux opérateurs hermitiens définis sur un ensemble D de l'espace de Hilbert avec $PD \subset D$ et $QD \subset D$; (II) $PQ - QP = -hiI$; (III) $P^2 + Q^2$ est décomposable dans D; (IV) (P, Q) est irréductible. Alors P et Q sont déterminés à une transformation unitaire près. Ce résultat est ensuite étendu au cas de f degrés de liberté. *M. Loève* (New York, N. Y.).

Plans y Sanz de Bremond, Antonio. Hilbert space. Revista Acad. Ci. Madrid **41**, 197–257 (1947). (Spanish)

The paper is expository. It deals with the elementary fundamental notions about orthonormal systems, linear manifolds, linear functionals, weak convergence, bounded linear operators, bounded selfadjoint operators, and projectors. *A. E. Taylor* (Cambridge, England).

Day, Mahlon M. Some characterizations of inner-product spaces. Trans. Amer. Math. Soc. **62**, 320–337 (1947).

The theorems of this paper give a number of necessary and sufficient conditions under which the norm in a real-linear or complex-linear normed space can be defined from an inner product. One of these conditions is

$$\|b_1+b_2\|^2+\|b_1-b_2\|^2=2(\|b_1\|^2+\|b_2\|^2)$$

for all vectors b_1, b_2 of norm one. This is a weakening of the condition given by P. Jordan and J. von Neumann [Ann. of Math. (2) **36**, 719–723 (1935)] which required that this be true for all vectors b_1, b_2. Another such condition is that, for all vectors b_i and linear functionals β_i,

$$(\beta_1+\beta_2)(b_1+b_2)=\|\beta_1+\beta_2\|\cdot\|b_1+b_2\|$$

whenever $\|b_i\|=\|\beta_i\|=\beta_i(b_i)=1$ for $i=1, 2$. The theorems are first established for real-linear normed spaces by showing that the conditions imply that every two dimensional cross-section of the unit sphere is an ellipse. The complex-linear case then follows from an elementary relationship between the complex-linear space and the real-linear space associated with it by using only real scalar multipliers. *R. S. Phillips* (Los Angeles, Calif.).

Mil'man, D. Characteristics of extremal points of regularly convex sets. Doklady Akad. Nauk SSSR (N.S.) **57**, 119–122 (1947). (Russian)

This note considers extreme points of regularly convex sets in the conjugate R^* of a complex Banach space. [See Krein and Šmulian, Ann. of Math. (2) **41**, 556–583 (1940); Krein and Mil'man, Studia Math. **9**, 133–138 (1940); these Rev. **1**, 335; **3**, 90, for real spaces.] Theorem 1. If T is a bounded set in R^* and T' is its w^*-closure (that is, closure in the product-space neighborhood topology), then the extreme points of K_T, the smallest regularly convex set containing T, lie in T'. Theorem 2. For a point f_0 of a bounded regularly convex set K to be extremal in K it is necessary and sufficient that for each w^*-closed subset E of K which does not contain f_0 the set K_E does not contain f_0. If $E \subseteq R^*$ is bounded and w^*-closed, define the T-frontier of E as the set of all f_0 in E such that for each w^*-neighborhood U of f_0 there is an x_U in R such that $\max \Re f(x_U)$ for $f \epsilon E$ is not attained in $E-U$. Theorem 3. If E is bounded and w^*-closed, then the T-frontier of E is precisely the w^*-closure of the set of extreme points of K_E.

If the set of maximal ideals of a commutative normed ring with unit is mapped homeomorphically in the standard way [Gelfand, Rec. Math. [Mat. Sbornik] N.S. **9**(51), 3–24 (1941); these Rev. **3**, 51] on a bounded closed subset E of R^*, the T-frontier becomes the "frontier" of Gelfand and Šilov [unpublished]. Theorem 3 yields directly theorem 4. The frontier of the set of all maximal ideals of R is the closure of the set of all extremal maximal ideals. *M. M. Day* (Urbana, Ill.).

Clarkson, James A. A characterization of C-spaces. Ann. of Math. (2) **48**, 845–850 (1947).

The problem of characterizing those Banach spaces B which are equivalent to a C-space (space of continuous functions on a compact Hausdorff space) has so far been treated only when additional algebraic postulates, not meaningful for all Banach spaces, are introduced [to the author's references, add: I. Gelfand, Rec. Math. [Mat. Sbornik] N.S. **9**(51), 3–24 (1941); M. Krein and S. Krein, C. R. (Doklady) Acad. Sci. URSS **27**, 427–430 (1940); these Rev. **2**, 222; **3**, 51]. The criterion here presented, involving Banach space terms only, is that C-spaces are just those Banach spaces in which the unit sphere is the intersection of a certain half-cone Q and its negative, where Q is so shaped that the intersection of two translated images $x+Q$, $y+Q$ is itself a translated image $Z+Q$, of the half-cone Q. In the proof of sufficiency, one forms a Banach lattice by setting $x \vee y = Z$ and, relative to the ordering, etc., thus introduced, establishes the hypothesis of S. Kakutani's characterization of C spaces. [One could, alternatively, prepare for an application of the theorem of Krein and Krein.] The geometrically intuitive necessity of the criterion is one of its main features. *R. Arens* (Los Angeles, Calif.).

Lonseth, A. T. The propagation of error in linear problems. Trans. Amer. Math. Soc. **62**, 193–212 (1947).

Let T, τ be linear on E to E, where E is a Banach space; if T is completely continuous, E need merely be a normed space. Consider (1) $(T+\tau)(x+\xi)=y+\eta$. Assume x satisfies (2) $Tx=y$. The terms τ, ξ and η are given interpretation in terms of errors. Assume T^{-1} exists and $\|T^{-1}\tau\|<1$. Then the Neumann expansion yields

$$(3) \qquad \|\xi\| \leqq (\|T^{-1}\|\|\eta\|+\|T^{-1}\tau\|\|x\|)/(1-\|T^{-1}\tau\|).$$

The paper interprets (3) in the norms of the c, C, l_p and L_p spaces. The results are obviously of use in practice. [With introduction of a parameter it would be clear that the results are those special cases of perturbations of resolvents for which the parameter values permit of the Neumann expan-

sion representation. Hence extension to other more general representations of the resolvent would seem to be of interest.]
D. G. Bourgin (Urbana, Ill.).

Hille, Einar. Sur les semi-groupes analytiques. C. R. Acad. Sci. Paris **225**, 445–447 (1947).

Let $\{T(\xi)|\xi>0\}$ belong to a semigroup of linear operators on the Banach space B to B. Under certain conditions an infinitesimal generator A can be defined by

$$A_x = L_{\xi\to 0}(T(\xi)-1)x/\xi.$$

Sufficient conditions are stated for the converse problem, namely that a closed distributive operator A on B to B be an infinitesimal generator of a semigroup $T(\xi, A)$. The properties of $T(\xi, A)$ are closely linked to the spectrum of A. The resolvent can be represented by the Laplace integral of $T(\xi)$ as the author has shown on several occasions and hence the following considerations are natural. The author defines an indicator function by

$$h(\theta) = L_{r\to\infty} \log \|T(re^{i\theta})\|/r,$$

$|\theta|<\alpha$, and states the correspondents of the Pólya theorems involving the conjugate indicator diagram. Here the singularities on the boundary of the diagram are points of the spectrum (since the function represented is the resolvent). Certain special cases are mentioned and a sort of converse argument for determining the conjugate diagram by way of $h(\theta)$ starting with A. *D. G. Bourgin* (Urbana, Ill.).

Povzner, A. On some general inversion formulas of Plancherel type. Doklady Akad. Nauk SSSR (N.S.) **57**, 123–125 (1947). (Russian)

Suppose that to each f in a dense linear subset H_0 of a Hilbert space H there corresponds a bounded linear operator A_f on H such that (1) $A_{\alpha f+\beta g}=\alpha A_f+\beta A_g$, (2) $A_fA_g=A_gA_f$, (3) if $f, g\epsilon H_0$, then $A_fg=A_gf\epsilon H_0$, (4) there exists an $e\epsilon H_0$ such that $A_e=E$ (the identity) and (5) there exists a linear subset $T_0\subset H_0$ such that, for $t\epsilon T_0$, A_t is Hermitian and each $h\epsilon H$ may be written in the form $h=t_1+it_2$, $t_1, t_2\epsilon T_0$. If a norm is defined in H_0 by $\|h\|_1=\|A_h\|$, then the completion of H_0 by this norm is a subset $\bar{R}$ of H which satisfies (1)–(5); with the multiplication $f\circ g=A_fg$, $\bar{R}$ is a normed ring. If T plays for $\bar{R}$ the role of T_0 for H_0, and if, for $r=t_1+it_2$ ($t_1, t_2\epsilon T$), the adjoint r^* is defined to be t_1-it_2, then, according to a result of Gelfand and Neumark, $\bar{R}$ is isomorphic to the ring $C(\mathfrak{M})$ of all continuous functions on the set $\mathfrak{M}$ of all maximal ideals of $\bar{R}$ [Rec. Math. [Mat. Sbornik] N.S. **12**(54), 197–213 (1943); these Rev. **5**, 147]. Theorem: there exists a measure ω on the Borel sets of $\mathfrak{M}$ such that, for $f, g, h\epsilon\bar{R}$, $(A_fg, h)=\int_{\mathfrak{M}}f(M)g(M)\overline{h(M)}d\omega(M)$; the mapping $f\to f(M)$ from $\bar{R}$ on $C(\mathfrak{M})$ may be extended to an isometric mapping from H on $L_2(\omega)$. This result is applied to prove Plancherel's theorem for a locally compact Abelian group G, by establishing a correspondence between the characters of G and the maximal ideals of a ring $\bar{R}$ generated by the above procedure. (If τ is the Haar measure of G, $H=L_2(\tau)$, and H_0 is the set of functions in $L_1(\tau)$ which vanish in a neighborhood of infinity, then the operators A_f, defined for $f\epsilon H_0$ and $g\epsilon H$ by A_fg = convolution of f and g, satisfy (1), (2), (3) and (5). By a rather artificial modification it is possible to satisfy (4) also, i.e., to introduce an identity into the system.) *P. R. Halmos* (Princeton, N. J.).

van Kampen, E. R., and Wintner, Aurel. On the asymptotic distribution of geodesics on surfaces of revolution. Časopis Pěst. Mat. Fys. **72**, 1–6 (1947). (English. Czech summary)

This note develops explicit formulas for the asymptotic distribution of a nonperiodic recurrent (in the sense of G. D. Birkhoff) geodesic on a surface of revolution. There are two cases to be considered, according as the geodesic is dense in a domain bounded by two parallel circles on the surface or the geodesic is dense on the whole surface. In the latter case the surface is of genus 1. The formulas are given by simple expressions in geometrical terms. An application is made to the path of a particle moving in a plane under the action of a central force. *G. A. Hedlund.*

Theory of Probability

Krishna Iyer, P. V. Random association of points on a lattice. Nature **160**, 714 (1947).

Points of k different colors are distributed over a lattice with probabilities $p_1, \cdots, p_k$, respectively. The author considers the distribution of the number of joins between points of different colors. He states without proof formulae for the mean and variance of this distribution in the case of one, two and three dimensional lattices. *H. B. Mann.*

Sawkins, D. T. A new method of approximating the binomial and hypergeometric probabilities. J. Proc. Roy. Soc. New South Wales **81**, 38–47 (1947).

Let p_r be the general term of the binomial distribution. The author writes $\log(p_{r+1}/p_r)=\Delta\log p_r$ and applies formulas such as $f'(x)=\Delta f(x)-\frac{1}{2}\Delta^2f(x)+\cdots$ to obtain formal expansions of p_r. The same method is used for the hypergeometric distribution $C_{r,Np}C_{n-r,Nq}C_{n,N}^{-1}$. *W. Feller.*

✶Robbins, Herbert. Some remarks on the inequality of Tchebychef. Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 345–350. Interscience Publishers, Inc., New York, 1948. $5.50.

Let C_n be the class of all random variables X which can be written in the form $X=X_1+\cdots+X_n$, where the X_i are mutually independent variables with identical distributions and where $E(X)=0$, $E(X^2)=1$. Let $\phi_n(t)=\sup\Pr(|X|\geqq t)$ for $X\epsilon C_n$. The author shows that $\phi_n(t)<t^{-2}$ for all $n>1$ and sufficiently large t (Chebyshev's inequality shows only $\phi_n(t)\leqq t^{-2}$), and that $t^2\phi_n(t)\to 1$ as $t\to\infty$. *W. Feller.*

Bochner, Salomon. Stochastic processes. Ann. of Math. (2) **48**, 1014–1061 (1947).

Details of results announced in an earlier paper [Proc. Nat. Acad. Sci. U. S. A. **32**, 259–261 (1946); these Rev. **8**, 215]. The author takes as a point of departure the randomization of finitely additive set functions $F(A)$ rather than point functions; that is, random variables are assigned to certain sets of a space instead of to certain values of a parameter. This has already been done in various special cases, but has not been done in a systematic way heretofore. Appropriate measures are constructed on the spaces of finitely additive set functions so that the author essentially deals with a finitely additive set function $F(A, \lambda)$ depending on a parameter λ, the probability parameter. The λ space has a probability measure defined in it and λ probabilities thus define set function probabilities. The author is par-

ticularly interested in integrals of the form $\int h(x)F(dx, \lambda)$ which are defined first for step functions $h(x)$ and then, by going to the limit, for a larger class of integrands. To justify the limiting process a measure of sets A is introduced and it is then possible to use an L_p norm in x space. In many cases it is possible to choose p so that when a sequence $h_n(x)$ converges (L_p norm) the corresponding sequence of stochastic integrals converges in λ measure. In these cases there is said to be L_p stability and the stochastic integral is defined by continuity for all integrands in L_p. The systematic use of such integrals, in special cases, apparently goes back to Wiener [cf. for example Paley and Wiener, Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 19, New York, 1934]. Note that the set functions need not be of bounded variation. The L_p stability of processes with independent increments is analyzed. If two processes are considered, correspondences between the $h(x)$'s of the two x spaces induce relations between the processes. For example, if the L_2 spaces of two L_2 stable processes have the same number of dimensions, there are isometries between them which map the processes on each other. As a particular case of this idea, the Fourier coefficients of an L_p stable process with $p \geqq 1$ are formed, $G_n(\lambda) = \int h_n(x)F(dx, \lambda)$, corresponding to an orthonormal set of h_n's. This takes the original process into the process of the coefficients, and conversely one can write $F(dx, \lambda)/dx \sim \sum_n G_n(\lambda)h_n(x)$; this symbolic equation can be interpreted in various ways. This idea is now exploited in several directions, for example to get solutions of the Poisson equation $\partial^2 g/\partial x^2 + \partial^2 g/\partial y^2 = -4\pi f$ replacing $f(x, y)$ by a distribution $dF(x, y)$, that is, replacing an integral involving f by one involving dF. This is done by introducing a finitely additive set function $F(A, \lambda)$ of such a type that for almost all λ there is a solution g. Finally the approach to stationary (wide sense) processes is given, obtaining the usual somewhat disguised spectral representations. *J. L. Doob.*

Kac, M. On the notion of recurrence in discrete stochastic processes. Bull. Amer. Math. Soc. 53, 1002–1010 (1947).

Let $x_0, x_1, \cdots$ be the random variables of a stationary stochastic process. The x_j's can assume only the values of a given finite or infinite sequence of numbers $a_1, a_2, \cdots$. The author proves the following theorems. (1) Each state is bound to recur, with probability 1, that is, if $\Pr\{x_0 = a_j\} > 0$, then $\lim_{n\to\infty} \Pr\{x_0 = a_j,\ x_1 \neq a_j,\ \cdots,\ x_n \neq a_j\} = 0$. (2) If $\lim_{n\to\infty} \Pr\{x_0 \neq a_j, \cdots, x_n \neq a_j\} = 0$, then if $x_0 = a_j$ and if θ is the expected value of the first subscript m for which $x_m = a_j$ once more, it follows that $\theta = 1/\Pr\{x_0 = a_j\}$. The first theorem is a reformulation in the language of probability of the classical Wiederkehrsatz, and the relation between the probability formulation and transformation theory formulation of such theorems is discussed. *J. L. Doob.*

Buchman, E. N. The problem of waiting time. Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 475–484 (1947). (Russian. English summary)

The paper is devoted to the problem of waiting times when k lines (counters) serve a traffic obeying a Poisson law. It is claimed that the analysis holds for arbitrary holding times, that is, also for non-Markov processes. Actually many assumptions are tacitly made which invalidate the analysis at least in its generality. *W. Feller.*

Mathematical Statistics

Rigg, F. A. Recent advances in mathematical statistics. J. Roy. Statist. Soc. (N.S.) 109, 395–450 (1946).

This continues the Society's bibliograpny, covering 1940–1942.

★Mather, K. Statistical Analysis in Biology. 2d ed. Interscience Publishers, Inc., New York, 1947. 267 pp. $5.00.

[The first edition appeared in 1943.] This text covers the same general ground as Fisher's well-known "Statistical Methods for Research Workers" [10th ed., Edinburgh, 1946], but with considerably more detailed explanation of the rationale of the methods. (This does not imply that mathematical proofs are given.) The general spirit of the book is indicated in a sentence on page 47: "The whole of the analysis of data really consists of their reduction to a state in which the question at issue can be formulated in terms of the ratio of two mean squares, or correspondingly of two root mean squares." A particularly valuable feature of the book is the careful discussion of single degrees of freedom and the isolation of their effects.

There are a few misstatements, due to failure to realize fully some of the mathematical limitations. For example, on page 212, it is stated that the variance of a maximum likelihood estimate is "always equal to the reciprocal of ni, where i is the amount of information," with no hint that this is a large sample property. Again, on page 231, there is evidence of confusion about sufficient estimates. Such misstatements are not likely to lead the user of the methods seriously astray, but will annoy the mathematical statistician. *C. P. Winsor* (Baltimore, Md.).

★Brownlee, K. A. Industrial Experimentation. Chemical Publishing Co., Inc., Brooklyn, N. Y., 1947. 151 pp. $3.75.

This book gives a concise presentation of the elementary statistical techniques and of the principal types of experimental design, with particular reference to their use in experiments on chemical manufacturing processes. No exposition of the mathematical theory is attempted.

W. G. Cochran (Raleigh, N. C.).

★Charlier, C. V. L. Elements of Mathematical Statistics. Also, L. v. Bortkiewicz, Table of Poisson's Frequency Function. Edited and translated by J. A. Greenwood. Cambridge, Massachusetts, 1947. iv+120 pp. Published by the translator, 25 Winthrop St., Brooklyn 25, N. Y. $3.00.

This book will interest the bibliophile and student of the history of mathematical statistics. It was a good book in 1910, but is now obsolete. *J. Wolfowitz.*

Fréchet, Maurice. Anciens et nouveaux indices de corrélation: errata. Econometrica 15, 374–375 (1947).

The paper appeared in the same vol., 1–30 (1947); these Rev. 8, 393.

v. Mises, R. On the asymptotic distribution of differentiable statistical functions. Ann. Math. Statistics 18, 309–348 (1947).

This paper is concerned with the asymptotic distribution, as $n\to\infty$, of a symmetric function $f = f(x_1, \cdots, x_n)$, where the x_i are independent random variables, such that $\Pr(x_i \leqq x) = V_i(x)$. Then f can be regarded as a function

$f=f\{S_n(x)\}$ of the repartition $S_n(x)$ of n sample values $x_1, \cdots, x_n$, where $nS_n(x)$ is the number of those x_i which are less than or equal to x. It is assumed that, in the space of all distribution functions $V(x)$, the function $f=f\{V(x)\}$ is defined in a convex domain D, including in particular (a) all possible repartitions $S_n(x)$, $n=1, 2, \cdots$, and (b) the functions $\bar{V}_n(x)=\sum_1^n V_i(x)/n$ for all sufficiently large n. Following Volterra, $f\{V(x)\}$ is said to possess a derivative f' of first order at a point $V_0(x)$ of D, if the relation

$$\frac{d}{dt}f\{V_0(x)+t[V(x)-V_0(x)]\}_{t=0} = \int_{-\infty}^{\infty} f'\{V_0(x), y\}[dV(y)-dV_0(y)]$$

holds for all points $V(x)$ of D, where f' depends on $V_0(x)$ and on a scalar variable y, but not on $V(x)$. Higher derivatives are correspondingly defined, the rth derivative depending on $V_0(x)$ and on r scalar variables $y_1, \cdots, y_r$.

The main theorem of the paper states that if, for all large n, the first $r-1$ derivatives of $f\{V(x)\}$ at the point $\bar{V}_n(x)$ vanish, while the rth derivative equals $\psi_n(y_1, \cdots, y_r)$, then, subject to certain conditions of regularity, the distributions of the random variables

$$A_n = n^{r/2}[f\{S_n(x)\}-f\{\bar{V}_n(x)\}]$$

and

$$B_n = \frac{n^{r/2}}{r!}\int\cdots\int \psi_n(y_1, \cdots, y_r)\prod_{i=1}^{r}[dS_n(y_i)-d\bar{V}_n(y_i)]$$

are asymptotically equal. Further, the asymptotic behaviour of the distribution is essentially determined by the functions $\bar{V}_n(x)$, $\psi_n(y_1, \cdots, y_r)$ and $\bar{U}_n(x, y)=\bar{V}_n(x)-\sum_1^n V_i(x)V_i(y)/n$ $(x\leqq y)$.

In the simplest case, $r=1$, the asymptotic distribution of A_n is normal, subject to further conditions of regularity not explicitly given. This covers the case of most ordinary statistics based on moments. For $r>1$, other types of asymptotic distributions appear. Thus in the case $r=2$, which is studied in detail, the normal distribution is replaced by a limiting distribution having the characteristic function $D(ui)^{-1}$, where $D(\lambda)$ is in general the Fredholm determinant of a symmetric kernel depending on $\bar{V}_n$, ψ_n and $\bar{U}_n$. The limiting distributions of χ^2 and ω^2 are particular cases of this type. *H. Cramér* (Stockholm).

Gumbel, E. J. The distribution of the range. Ann. Math. Statistics 18, 384–412 (1947).

This paper brings the author's studies on extremes and ranges [the preceding one was published in the same Ann. 17, 78–81 (1946); these Rev. 7, 464] to a kind of conclusion. He considers unlimited symmetrical distributions of "exponential type" and deduces the asymptotical distribution of the "reduced" sample range $R=\alpha_n(w-2u_n)$, where w is the ordinary range, and α_n, u_n are parameters depending on the initial distribution and the sample size n. The result is expressed in terms of Hankel functions. Numerous tables and graphs are given and some generalizations sketched. [Cf. the reviewer's article in Biometrika 34, 111–119 (1946); these Rev. 8, 395, where the same purpose (for the normal and related cases) was reached by means of the transformation $z=2n\Phi(-\frac{1}{2}w)$; z corresponds asymptotically to $2e^{-R/2}$. A comparison with Hartley's tables shows that the accuracy of both methods, for a normal initial distribution and $n=20$, is about the same; for small ranges, however, Gumbel's figures show somewhat greater discrepancies.] *G. Elfving* (Helsingfors).

Chakrabarti, M. C. On a special case of the distribution law of the mean square successive difference. Bull. Calcutta Math. Soc. 39, 15–18 (1947).

The probability function of δ^2, $(n-1)\delta^2=\sum_{i=1}^{n-1}(x_{i+1}-x_i)^2$, when the variates are distributed normally with mean 0 and variance σ^2, is derived for $n=3$ by inversion of the characteristic function of δ^2. The result has been proved otherwise by von Neumann, Kent, Bellinson and Hart [Ann. Math. Statistics 12, 153–162 (1941); these Rev. 3, 7]. *L. A. Aroian* (New York, N. Y.).

Egudin, G. I. On the stability of some very general classes of statistics. Doklady Akad. Nauk SSSR (N.S.) 57, 115–117 (1947). (Russian)

The author proves the well-known result that a polynomial in the sample moments is consistent (Russian: stable). *J. Wolfowitz* (New York, N. Y.).

Weaver, C. L. A simple analytic proof of a general χ^2 theorem. Amer. Math. Monthly 54, 529–533 (1947).

Let y_i, $i=1$ to n, independent variables, be distributed with mean $a_0+a_1t_{1i}+a_2t_{2i}+\cdots+a_lt_{li}$ and variance σ_i^2; the t's may be functionally related but are assumed to be linearly independent. In the set of $l+1$ parameters, l_1 values are known (the set a_r) but the other l_2 values are unknown (the set a_s), $l_2<n$. The following general χ^2 theorem is proved analytically. If the set $\hat{a}_s$ is the set of maximum likelihood estimates of the set a_s then $P=\sum_i\sigma_i^{-2}\{y_i-\sum_r a_rt_{ri}-\sum \hat{a}_st_{si}\}^2$ is distributed as χ^2 with $n-l_2$ degrees of freedom, and independently of the set $\hat{a}_s$ which are jointly distributed in the l_2-variate normal distribution $C_1e^{-\frac{1}{2}Q}\prod_s d\hat{a}_s$ with

$$Q=\sum_i\sigma_i^{-2}\{\sum(\hat{a}_s-a_s)t_{si}\}^2.$$

This Q is distributed independently of P as χ^2 with l_2 degrees of freedom. If $R=\sum_i\sigma_i^{-2}\{y_i-\sum_j a_jt_{ji}\}^2$, then $R=P+Q$ and R is distributed as χ^2 with n degrees of freedom. *L. A. Aroian* (New York, N. Y.).

Hastings, Cecil, Jr., Mosteller, Frederick, Tukey, John W., and Winsor, Charles P. Low moments for small samples: A comparative study of order statistics. Ann. Math. Statistics 18, 413–426 (1947).

The authors give in this paper tables illustrating statistics based on ordered observations, so-called systematic statistics [cf. Mosteller, same Ann. 17, 377–408 (1946); these Rev. 8, 477]. The paper gives means, variances and covariances for samples of size less than or equal to 10 from (1) the normal distribution, (2) the rectangular distribution and (3) a distribution with long tails, defined by $r(u)=(1-u)^{-1/10}-u^{-1/10}$, where u has rectangular distribution in the interval [0, 1] and $x=r(u)$ is the variable considered. The tables, moreover, give approximate values for the means, variances and covariances for the different distributions, computed from asymptotic formulae. The methods applied and the accuracy acquired are further discussed. *K. R. Buch* (Copenhagen).

Brown, George W. Discriminant functions. Ann. Math. Statistics 18, 514–528 (1947).

Brown, George W. On small-sample estimation. Ann. Math. Statistics 18, 582–585 (1947).

TOPOLOGY

Kagno, I. N. Desargues' and Pappus' graphs and their groups. Amer. J. Math. **69**, 859–862 (1947).

If a graph with V vertices and E edges can be embedded into a surface of characteristic K to form a map of F_3 triangles, F_4 quadrangles, and so on, we have

$$2E=3F_3+4F_4+\cdots\geqq 3(F_3+F_4+\cdots),$$

whence $K=V-E+(F_3+F_4+\cdots)\leqq V-E/3$. It follows that no graph of degree 6 (for which $V=E/3$) can be embedded into the projective plane (for which $K=1$). The author verifies this in two special cases: graphs derived from the configurations of Desargues and Pappus by naming an edge for every pair of points belonging to a line of the configuration. He makes the observation that the former (which can be embedded into the surface having $K=-1$) is the complement of the Petersen graph; and he deduces that its group, like that of the configuration, is the symmetric group of order $5!$. He just misses the analogous observation that the Pappus graph (which can be embedded into the torus, $K=0$) is the complement of three separate triangles. This would have simplified his proof that the group of the Pappus graph is the direct product of four symmetric groups, each of order $3!$. *H. S. M. Coxeter* (Toronto, Ont.).

Ridder, J. Einige Anwendungen des Dualitätsprinzips in topologischen Strukturen. Nederl. Akad. Wetensch., Proc. **50**, 731–740 = Indagationes Math. **9**, 341–350 (1947).

The author discusses some simple consequences (concerning the characterization of interior by a single relation, boundaries, topological separation axioms, etc.) of the duality (in a Boolean algebra) between certain (postulated) concepts of closure and interior. *P. R. Halmos.*

Monteiro, António. La notion de fermeture et les axiomes de séparation. Anais Fac. Ci. Pôrto **26**, 193–203 (1941).

The paper also appeared in Portugaliae Math. **2**, 290–298 (1941); these Rev. **4**, 87.

Monteiro, António. Introduction to the study of the notion of a continuous function. Anais Fac. Ci. Pôrto **28**, 225–371 (1943). (Portuguese)

The paper also appeared as Publ. Centro Estudos Mat. Fac. Ci. Pôrto, Inst. para a Alta Cultura, Lisbon, no. **8** (1944); these Rev. **6**, 94.

Denjoy, Arnaud. Définition intrinsèque, non pas ordinale, de l'arc et de la dendrite. C. R. Acad. Sci. Paris **225**, 773–776 (1947).

The author obtains a characterization of the dendrite as a continuum E every subcontinuum of which contains uncountably many cut points of E and characterizes the simple arc as a dendrite every cut point of which separates it into exactly two components. He is apparently unaware of the fact that this characterization of dendrites was published by R. L. Moore in 1923 [Proc. Nat. Acad. Sci. U. S. A. **9**, 101–106]. The arc characterization follows readily from it.
G. T. Whyburn (Charlottesville, Va.).

Whyburn, G. T. On locally simple curves. Bull. Amer. Math. Soc. **53**, 986–992 (1947).

Simple topological characterizations are obtained of metric continua that admit locally simple representations on the circle [previously studied by Morse and Heins, Proc. Nat. Acad. Sci. U. S. A. **31**, 299–301, 302–306 (1945); Ann. of Math. (2) **46**, 600–624, 625–666 (1945); **47**, 233–273 (1946); these Rev. **7**, 57; **8**, 21]. For example, such a continuum is characterized by the fact that it is the sum of a finite number of doubly extensible arcs. If the continuum is cyclic, it is characterized by being the sum of a finite number of simple closed curves. If it is a boundary curve, it is characterized by the fact that it has only a finite number of nodes and no end points. Among the topological by-products is a simple proof of the well-known cyclic connectivity theorem [see Whyburn, Analytic Topology, Amer. Math. Soc. Colloquium Publ., v. 28, New York, 1942, p. 79; these Rev. **4**, 86]. *R. L. Wilder* (Ann Arbor, Mich.).

Dowker, C. H. An imbedding theorem for paracompact metric spaces. Duke Math. J. **14**, 639–645 (1947).

L'auteur démontre qu'un espace métrique paracompact est homéomorphe à un sous-espace d'un espace de Hilbert (généralisé, c'est-à-dire où l'ensemble des indices des coordonnées a une puissance quelconque), et inversement que tout sous-espace d'un espace de Hilbert est paracompact (de sorte que les espaces de Hilbert sont des "espaces universels" pour les espaces métriques paracompacts). La première partie de cet énoncé se démontre assez facilement, mais la seconde résulte d'une analyse subtile de la structure des espaces métriques paracompacts, au cours de laquelle l'auteur donne entre autres une démonstration directe simple du fait que tout espace métrique séparable est paracompact, et prouve que toute réunion dénombrable de sous-espaces paracompacts d'un espace métrique est un sous-espace paracompact; le résultat final découle alors de cette analyse et d'un choix convenable d'une base de la topologie d'un espace de Hilbert. *J. Dieudonné* (Nancy).

Koszul, Jean-Louis. Sur les opérateurs de dérivation dans un anneau. C. R. Acad. Sci. Paris **225**, 217–219 (1947).

A ring A is considered with a derivation operator D having the formal properties of the coboundary operator of algebraic topology. It is further assumed that a sequence of ideals B^p (p integer) is given with the following properties: $\cup B^p=A$, $\cap B^p=(0)$, $D:B^p\rightarrow B^p$. The following definitions are then made: $C_r^p=B^p\cap D^{-1}(B^{p+r})$, $D_r^p=B^p\cap D(B^{p-r})$, $E_r^p=C_r^p/(D_{r-1}^p+C_{r-1}^{p+1})$, $E_r=\sum_p E_r^p$. Each E_r is then given the structure of a ring with a derivation and it turns out that E_{r+1} is the cohomology ring of E_r. The sequence $\{E_r\}$ gives in a certain sense an approximation to the cohomology ring of A. These purely algebraic considerations are an abstraction of definitions made by Leray [same C. R. **222**, 1366–1368, 1419–1422 (1946); these Rev. **8**, 49] in connection with his cohomology ring of a continuous mapping. The author also promises other topological applications [see the following review]. *S. Eilenberg* (New York, N. Y.).

Koszul, Jean-Louis. Sur l'homologie des espaces homogènes. C. R. Acad. Sci. Paris **225**, 477–479 (1947).

Let G be a compact connected Lie group, U a connected subgroup, p the natural mapping of G onto the homogeneous space $W=G/U$. Applying the considerations of an earlier note [see the preceding review] to the ring of left invariant differential forms of G, the author is led to the determination of the cohomology ring of p. Many applications are announced. For instance: if U is one-dimensional and homologous to zero (as a 1-cycle) in G then $G(t)(1+t^2)=W(t)(1+t^1)$, where $G(t)$ and $W(t)$ are the Poincaré polynomials of G and W. *S. Eilenberg* (New York, N. Y.).

Hu, Sze-tsen. On spherical mappings in a metric space. Ann. of Math. (2) **48**, 717–734 (1947).

Abe [Jap. J. Math. 16, 169–176 (1939); these Rev. **2**, 71] defined a group $\kappa_0^n(Y)$ of a space Y which included the fundamental group $\pi_1(Y)$, the nth homotopy group $\pi_n(Y)$ and the operations of π_1 on π_n. In the present paper the Abe definition is generalized to provide a group $\kappa_r^n(Y)$ for each $0 \leqq r < n$. When $r > 0$, the direct sum decomposition $\kappa_r^n \approx \pi_{r+1} + \pi_n$ is proved. This suggests that the operations of π_1 on π_n have no nontrivial higher dimensional analogue.

N. E. Steenrod (Princeton, N. J.).

Hu, Sze-Tsen. On homotopy and deformation retracts. Proc. Cambridge Philos. Soc. **43**, 314–320 (1947).

Ein Satz von Hurewicz [Nederl. Akad. Wetensch., Proc. **38**, 112–119 (1935)] besagt: Wenn alle Homotopiegruppen eines zusammenhängenden Kompaktums K, welches ein absoluter Umgebungsretrakt ist, verschwinden, so ist K in sich zusammenziehbar. Die vorliegende Arbeit behandelt den entsprechenden "relativen" Fall. Es sei Y ein separabler metrischer Raum, Y_0 eine abgeschlossene Teilmenge von Y, beides zusammenhängende absolute Umgebungsretrakte. Der Raum Y wird "n-asphärisch modulo Y_0" genannt, $n > 1$, wenn die n-te relative Homotopiegruppe von $Y \bmod Y_0$ gleich 0 ist; $n = 1$, wenn jede stetige Abbildung eines Kreises in Y sich in Y_0 deformieren lässt. Satz (1): Y sei r-asphärisch mod Y_0 für $1 \leqq r \leqq n$; f sei eine Abbildung eines Kompaktums X in Y, für welche $\dim f^{-1}(Y - Y_0) \leqq n$ ist. Dann ist f einer Abbildung von X in Y_0 homotop, und zwar so, dass bei der Deformation f_t $(0 \leqq t \leqq 1)$ durchwegs $f_t f^{-1}(Y_0) \subset Y_0$ ist; man kann sogar erreichen, dass in $f^{-1}(Y_0)$ durchwegs $f_t = f$ ist. Daraus ergibt sich (2): Ist Y kompakt und r-asphärisch mod Y_0 für $1 \leqq r \leqq \dim (Y - Y_0)$, so ist Y_0 Deformationsretrakt von Y; es gibt sogar, was damit äquivalent ist, eine Deformation von Y in Y_0, bei welcher die Punkte von Y_0 durchwegs fest bleiben. Für (endliche oder unendliche) Polyeder wurde (2) von J. H. C. Whitehead [Proc. London Math. Soc. (2) **45**, 243–327 (1939)] bewiesen. Der Beweis von (1) beruht auf dem Erweiterungssatz von Kuratowski [Fund. Math. **24**, 259–268 (1935)].

B. Eckmann (Lausanne).

Wang, Hsien-Chung. Some examples concerning the relations between homology and homotopy groups. Nederl. Akad. Wetensch., Proc. **50**, 873–875 = Indagationes Math. **9**, 384–386 (1947).

Consider M_1, the Cartesian product of an n-dimensional sphere with an $(n+2)$-dimensional projective space, and M_2, the Cartesian product of an n-dimensional projective space with an $(n+2)$-dimensional sphere. Then $H^n(M_1)$ is not isomorphic to $H^n(M_2)$ and $H^{n+1}(M_1)/\Sigma^{n+1}(M_1)$ is not isomorphic to $H^{n+1}(M_2)/\Sigma^{n+1}(M_2)$ (where H denotes homology group with integer coefficients and Σ denotes the subgroup of spherical homology classes). Nevertheless M_1 and M_2 have the same homotopy groups in every dimension (and they even vanish for dimensions $2, 3, \cdots, n-1$). Thus the homology groups H and the spherical quotient groups H/Σ of a space are not determined by its homotopy groups. S. Eilenberg pointed out to the reviewer that, for odd n, the example proves more, inasmuch as the homotopy groups are then operator isomorphic (with respect to the fundamental group).

R. H. Fox (Princeton, N. J.).

Hamilton, O. H. A fixed point theorem for upper semi-continuous transformations on n-cells for which the images of points are non-acyclic continua. Duke Math. J. **14**, 689–693 (1947).

A multi-valued transformation T of a space A into a space B is called continuous (or upper semi-continuous) if the relation $\lim p_i = p$ in A implies $\operatorname{Lim} T(p_i) = T(p)$ (or $\operatorname{Lim} T(p_i) \subset T(p)$) in B. The case is considered when $A = B = I_n$ is an n-cell and when, for each $p \epsilon I_n$, the set $T(p)$ is a topological $(n-1)$-sphere. The existence of a point p with $p \epsilon T(p)$ is proved under either of the following conditions: (a) T is continuous, (b) T is upper semi-continuous and each $T(p)$ contains in its interior a sphere of fixed diameter d. Theorems are also proved concerning the structure of the sets of points $\{p\}$ for which p is outside, on, or inside $T(p)$.

S. Eilenberg (New York, N. Y.).

GEOMETRY

Conway, A. W. Application of quaternions to rotations in hyperbolic space of four dimensions. Proc. Roy. Soc. London. Ser. A. **191**, 137–145 (1947).

The idea of representing a point (x_0, x_1, x_2, x_3) in Euclidean 4-space by a quaternion $x = x_0 + x_1 i + x_2 j + x_3 k$ was developed by Hathaway [Bull. Amer. Math. Soc. **4**, 54–57 (1897)] and Stringham [Trans. Amer. Math. Soc. **2**, 183–214 (1901); see also Coxeter, Amer. Math. Monthly **53**, 136–146 (1946); these Rev. **7**, 387]. The present paper is an attempt to use the same representation for a point in Minkowskian 4-space (not hyperbolic 4-space as the title would make one expect). The interval between two points x and y is now the scalar part of $(x-y)^2$. After some intricate manipulations the author concludes that the general Lorentz transformation is the product of two commutative "rotations" in completely orthogonal planes: an ordinary rotation in a separation plane (Euclidean plane), and a hyperbolic rotation in an inertia plane (Minkowskian plane). When expressed in terms of hyperbolic 3-space in the manner suggested by A. A. Robb [Geometry of Time and Space, Cambridge University Press, 1936, p. 406], this result simply means that any displacement can be regarded as a screw-displacement: rotation about a line combined with translation along the same line.

H. S. M. Coxeter (Toronto, Ont.).

Hadwiger, H. Bemerkung über vierdimensionale reguläre Polytope und Quaternionen. Mitt. Naturforsch. Ges. Bern **1942**, LVIII–LX (1943).

A set of vectors $\mathbf{a}_1, \cdots, \mathbf{a}_n$ in Euclidean s-space is called a eutactic star if $\sum \mathbf{a}_j \mathbf{a}_j \cdot \mathbf{x} = \lambda \mathbf{x}$ with the same λ for every $\mathbf{x}$, namely $\lambda = \sum \mathbf{a}_j \cdot \mathbf{a}_j / s$. In particular, the vectors from the center of a regular polytope to its vertices form a eutactic star [Schläfli, "Theorie der vielfachen Kontinuität," Denkschr. Schweiz. Naturforsch. Gesellschaft **38** (1901), p. 138; Coxeter, Regular Polytopes, London, 1948, pp. 251, 261]. If $s = 2$ or 4 or 8, we may represent each vector by a complex number or quaternion or Cayley number, and the inner product $\mathbf{a} \cdot \mathbf{b}$ takes the form $(\bar{\mathbf{a}}\mathbf{b} + \bar{\mathbf{b}}\mathbf{a})/2$, so that the above relation becomes $\sum \mathbf{a}_j (\bar{\mathbf{a}}_j \mathbf{x} + \bar{\mathbf{x}} \mathbf{a}_j)/2 = \sum \mathbf{a}_j \bar{\mathbf{a}}_j \mathbf{x} / s$, whence $\sum \mathbf{a}_j \bar{\mathbf{x}} \mathbf{a}_j / \sum \mathbf{a}_j \bar{\mathbf{a}}_j = (2s^{-1} - 1)\mathbf{x}$. The author obtains this result with $\mathbf{x} = 1$ and $s = 4$; but the extension to $s = 8$ is just as easy. In particular, the sum of the squares of the 24 quaternion units is -12, and the sum of the squares of the 240 units in the domain of integral ~~quaternions~~ Cayley numbers is -180. A

geometric interpretation is suggested by the fact that $-\mathbf{a}\bar{\mathbf{x}}\mathbf{a}/\mathbf{a}\bar{\mathbf{a}}$ is the image of the vector $\mathbf{x}$ by reflection in the hyperplane perpendicular to $\mathbf{a}$ [Witt, Abh. Math. Sem. Hansischen Univ. **14**, 289–322 (1941), p. 308; Coxeter, Duke Math. J. **13**, 561–578 (1946), p. 570; these Rev. **3**, 100; **8**, 370]. *H. S. M. Coxeter* (Toronto, Ont.).

Varga, O. Über die Lösung differentialgeometrischer Fragen in der nichteuklidischen Geometrie unter gleichzeitiger Verwendung homogener und inhomogener Koordinaten. Hungarica Acta Math. 1, 35–52 (1947).

Let a curve in n-dimensional space be regarded as the limiting form of a polygon $P_0P_1P_2\cdots$, so that its osculating k-space is the limiting form of the k-space $P_0\cdots P_k$. The author expresses the kth curvature (i.e., the arc-rate of turning of the osculating k-space) as the limit of

$$\frac{(k+1)^2(P_0\cdots P_{k+1})(P_1\cdots P_k)}{k(P_0P_{k+1})(P_0\cdots P_k)(P_1\cdots P_{k+1})},$$

where $(P_0\cdots P_k)$ is the content of the k-dimensional simplex $P_0\cdots P_k$. He finds that this holds in non-Euclidean space as well as in Euclidean. [Cf. Egerváry and Alexits, Comment. Math. Helv. **13**, 257–276 (1941); these Rev. **3**, 185.] In the case of elliptic space, he also expresses the $(k+1)$th curvature in the form

$$\tfrac{1}{2}(\sec^2 r_k)'(\sec^2 r_{k+1}-\sec^2 r_k)^{-\frac{1}{2}}(\sec^2 r_k-\sec^2 r_{k-1})^{-\frac{1}{2}},$$

where r_k is the radius of the kth sphere of curvature (i.e., the limiting form of the circumsphere of the simplex $P_0\cdots P_{k+1}$), and the prime indicates differentiation with respect to the arc of the curve. *H. S. M. Coxeter.*

Egerváry, E. On a generalisation of a theorem of Sylvester. Hungarica Acta Math. **1**, 53–57 (1947).

The center of gravity of the convex polyhedral region of six vertices (the most general case is an octahedron) is determined as follows. Construct the common perpendiculars of the three diagonals taken in pairs. Then the intersection of the perpendicular bisecting planes is the cross-center C. The center of gravity of six equal weights placed at the vertices is the mid-center M. Then the center of gravity of the volume is given by the vector equation $G=(3M-C)/2$. The same theorem for the special case of the truncated triangular pyramid had been demonstrated previously by Sylvester. *M. Goldberg.*

Thébault, Victor. Theorem on the trapezoid. Amer. Math. Monthly **54**, 537–538 (1947).

Blanchard, René. Les sphères de Hagge d'un polyèdre à sommets cosphériques. C. R. Acad. Sci. Paris **225**, 980–982 (1947).

Cavallaro, Vincenzo G. Sur les triangles ayant singulière la distance du centre de l'ellipse de Brocard au centre du cercle des neuf points. Anais Fac. Ci. Pôrto **25**, 129–140 (1940).

Cavallaro, Vincenzo G. Sur les distances mutuelles des points remarquables de la géométrie du triangle. Anais Fac. Ci. Pôrto **29**, 5–10 (1944).

Cavallaro, Vincenzo G. Formules brocardiennes pour le triangle singulier ayant les côtés en progression arithmétique. Anais Fac. Ci. Pôrto **29**, 11–14 (1944).

Toscano, Letterio. Sui triangoli armonici. Anais Fac. Ci. Pôrto **28**, 73–83 (1943).

Queiroz, Augusto, and Rios de Souza, Jayme. Circles which are projected into circles. Anais Fac. Ci. Pôrto **29**, 177–211 (1945). (Portuguese)

Godeaux, Lucien. Sur la théorie des réciprocités du plan. Anais Fac. Ci. Pôrto **26**, 158–159 (1941).

Addendum to a paper in the same Anais **22**, 211–215 (1937).

Maurin, Jacques. Géométrie descriptive à quatre dimensions. C. R. Acad. Sci. Paris **225**, 560–562 (1947).

On sait que la description d'un être géométrique aux environs d'un de ses points se réduit à la détermination des distances de l'être du plan tangent au point envisagé. La description graphique est donc un problème de la géométrie descriptive des hyperespaces, si l'être géométrique est de trois dimensions au moins. C'est pourquoi l'auteur fonde la géométrie descriptive à quatre dimensions en représentant un tel espace dans un espace de deux dimensions. Cette méthode est susceptible d'une généralisation à des espaces de dimensions quelconques. L'essentiel de cette représentation est qu'un point quelconque de l'espace est représenté sur trois points situés sur une droite de rappel orthogonale à une certaine droite. L'idée qui conduit à cette représentation est la suivante: on rapporte l'espace de quatre dimensions à un système de coordonnées rectangulaires. En choisissant arbitrairement un des quatre axes, on pose les trois plans de deux dimensions sur les autres axes. En faisant la projection du point sur ces trois plans et en rabattant d'une façon convenable les trois plans autour de l'axe commun sur le plan de figure, les trois distances de projection de l'axe commun sont situées sur une droite de rappel orthogonale à l'axe commun. La quatrième coordonnée est l'abscisse sur l'axe commun. A propos de l'exemple de la construction du point unique commun à deux plans de deux dimensions, l'auteur donne la détermination des autres variétés linéaires. Cette détermination est la méthode des traces, bien connue dans la géométrie descriptive. *O. Varga* (Debrecen).

Fukuzawa, S. Set of shadows. Tensor **4**, 77–80 (1941). (Japanese)

Under the postulate that for any n elements their images on one another are considered always as new elements (for example n mirrors and their images on one another), we get an infinite set of elements which contains all images of every element on one another. This paper studies the properties of this set. *A. Kawaguchi* (Sapporo).

Algebraic Geometry

Pedoe, D. On a new analytical representation of curves in space. Proc. Cambridge Philos. Soc. **43**, 455–458 (1947).

The associated form $F(u^0, u^1)$ of an algebraic curve C is also a form $G(\cdots, p_{ij}, \cdots)$ in the line coördinates [Hodge]. The author observes that if (α) is a general point of the space then $G(\cdots, x_i\alpha_j-x_j\alpha_i, \cdots)=0$ gives the equation of the cone which projects C from (α) and hence the equations of C itself. [Contrary to the author's belief, his unfinished

proof of the above theorem of Hodge can be completed by elementary methods: write the equation of the Grassmannian quadric in the form $x_0^2-x_1^2+x_2^2-x_3^2+x_4^2-x_5^2=0$, where $x_0=p_{01}+p_{23}$, $x_1=p_{01}-p_{23}$, etc., and observe that the formula at the bottom of page 457 yields for $F(u^0, u^1)$ an expression of the form $(A+Bx_5)/(x_0+x_1)^g$, where A and B are forms in $x_0, x_1, \cdots, x_4$, and also (by interchanging p_{01} and p_{23}) a similar expression with $(x_0-x_1)^g$ in the denominator. Use the fact that 1 and x_5 are linearly independent over the field $k(x_0, x_1, \cdots, x_4)$.] *O. Zariski.*

Turri, Tullio. Sulle trasformazioni antibirazionali involutorie del piano. Rend. Sem. Fac. Sci. Univ. Cagliari **15** (1945), 189–192 (1947).

Sunto: Si dimostra che ogni trasformazione antibirazionale involutoria del piano è birazionalmente il prodotto di una involuzione piana reale del secondo ordine per la trasformazione di coniugio.

Aus dem Text geht nicht eindeutig hervor, was dabei unter "birazionalmente" zu verstehen ist: Soll es heissen: ". . . ist in der Gruppe der Cremona-Transformationen gleich einem Produkt . . ." oder aber "durch eine Cremona-Transformation in ein solches Produkt transformierbar?" Für die antibirationale Involution:

$$\begin{aligned} x' &= \rho^2\bar{x}\bar{y}\bar{z}+\bar{x}^2\bar{z}-\tfrac{3}{2}\rho\bar{x}\bar{y}, \\ y' &= \bar{y}^2\bar{z}+\rho\bar{x}\bar{y}\bar{z}-\tfrac{3}{2}\rho^2\bar{x}\bar{y}^2, \\ z' &= -\rho\bar{y}^2\bar{z}-\rho\bar{x}^2\bar{y}+\tfrac{3}{2}\bar{x}\bar{y}\bar{z}, \qquad \rho^3=1, \end{aligned}$$

treffen freilich beide Aussagen nicht zu. Beim Beweis stützt sich Verf. auf den Satz, dass von zwei vertauschbaren Involutionen entweder beide denselben Grad haben oder eine linear sei. Dieser Satz trifft z.B. für $x_s'=1/x^{2r}$, $y'=y$, und $x'=\bar{x}$, $y'=\bar{y}-i\bar{x}^n$ nicht zu. Die in den beiden Bemerkungen gemachten Aussagen treffen nur für die dort angeführten Beispiele zu und dürfen nicht verallgemeinert werden.
O.-H. Keller (Dresden).

Bompiani, E. Varietà prodotto topologico di spazi multipli. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 493–497 (1947).

L'auteur étudie les variétés $V[m_1^{\mu_1}\cdots m_k^{\mu_k}]$ produit topologique des espaces linéaires $S_{m_1}, \cdots, S_{m_k}$ comptés avec les multiplicités $\mu_1, \cdots, \mu_k$ (l'ordre n'important pas). Elles contiennent comme cas particulier les variétés de Veronese $V[m^\mu]$ et les variétés de Segre $V[m_1^1\cdots m_k^1]$. Exemples: la surface représentative d'une droite simple et d'une droite double est la réglée R^4 de S_5; la surface représentative de deux droites doubles est la surface F^8 de S_8 de del Pezzo [Rend. Circ. Mat. Palermo 1, 241–271 (1887)]. La variété générale étudiée a pour dimension $m=m_1+\cdots+m_k$; elle appartient à un espace de dimension $C_{m_1+\mu_1}^{m_1}\cdots C_{m_k+\mu_k}^{m_k}-1$; son ordre est $(m!/m_1!\cdots m_k!)\mu_1^{m_1}\cdots\mu_k^{m_k}$. Elle est rationnelle et contient k systèmes de variétés de Veronese $V[m_i^{\mu_i}]$ de dimensions $m-m_i$ tels que par tout point passe une variété de chaque système: les espaces tangents déterminent complètement celui de la variété donnée. L'auteur étudie ensuite la variété duale, les propriétés de la variété considérée comme section d'une variété de Segre.
L. Gauthier (Nancy).

Godeaux, Lucien. Remarques sur une involution appartenant à une surface du cinquième ordre. Anais Fac. Ci. Pôrto **25**, 193–198 (1940).

Godeaux, Lucien. Sur les surfaces circonscrites à une surface cubique. Anais Fac. Ci. Pôrto **30**, 11–21 (1945).

Godeaux, Lucien. Remarques sur les systèmes linéaires de courbes tracées sur une surface algébrique et sur un théorème de Picard. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 403–410 (1947).

Burniat, Pol. Surfaces canoniques multiples abéliennes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 435–441 (1947).

Ales, Maria. Forme canoniche per le forme differenziali triple in cinque variabili ed un teorema sugli spazi doppi a cinque dimensioni contenenti forme algebrico-differenziali triple di prima specie. Rend. Circ. Mat. Palermo **62**, 286–288 (1940).

Ales, Maria. Alcune osservazioni sulle rappresentazioni parametriche. Rend. Circ. Mat. Palermo **63**, 91–92 (1942).

Ales, Maria. Osservazioni intorno agli invarianti proiettivi di terne di elementi curvilinei. Rend. Circ. Mat. Palermo **63**, 111–112 (1942).

The following theorem was proved by P. Buzano [Boll. Un. Mat. Ital. (2) **3**, 201–207 (1941); these Rev. **3**, 182; **7**, 620] for the case $n=2$: if the projective invariants of a triple of curvilinear elements of order $n-1$ in S_n have values independent of the choice of these elements from among the elements of a curve C in S_n, then C is a rational normal curve. The present note points out that Buzano's proof can be applied equally well to the general case.
R. J. Walker (Ithaca, N. Y.).

Ales, Maria. Intorno ad una proprietà caratteristica delle varietà trasformabili razionalmente nel prodotto topologico di due curve algebriche. Rend. Circ. Mat. Palermo **62**, 382–384 (1941).

Cherubino, Salvatore. Sul criterio di equivalenza. Rend. Circ. Mat. Palermo **62**, 369–376 (1941).

Differential Geometry

Cox, J. F. The doubly equidistant projection. Bull. Géodésique N.S. **1946**, no. 2, 74–76 (1946).

The deformation characteristics of the doubly equidistant projection of a sphere on a plane are discussed by reference to the differential indicatrix in the neighborhood of an arbitrary point. It is shown that zero deformation occurs on the axis joining the reference points, and elsewhere a correlation exists between the indicatrix and certain great circle representations on the projection. *N. A. Hall.*

Vázquez García, Roberto, and Barros Sierra, Javier. Theorems on geodesic circles and the Gaussian curvature. Bol. Soc. Mat. Mexicana **3**, 45–56 (1946). (Spanish)

Birkhoff, Garrett. Formulation of a conjecture of George D. Birkhoff by means of an integral equation. Bol. Soc. Mat. Mexicana **3**, 57–60 (1946). (Spanish)

Barajas, Alberto, and Vázquez, Roberto. A theorem related to a conjecture of G. D. Birkhoff. Bol. Soc. Mat. Mexicana **3**, 61–64 (1946). (Spanish)

[In the original the author of the second paper appears as Garret Birkhoff.] Some elementary results related to the conjecture of G. D. Birkhoff that if a surface has the prop-

erty that, for some fixed constant $a>0$, all geodesic circles of radius a have the same area, the surface must have constant Gaussian curvature. *G. A. Hedlund.*

Bouligand, Georges. Sur la topologie restreinte du second ordre. Revue Sci. **85**, 282–285 (1947).

In this note the author gives a more detailed discussion of a program started in an earlier note [C. R. Acad. Sci. Paris **224**, 1261–1263 (1947); these Rev. **8**, 531]. The paper is concerned with an extension of differential geometry, in which the group of rigid motions of 3-space is replaced by the group of homeomorphisms of class C^2, and tangent planes and osculating quadrics of a surface S are replaced by general surfaces. A typical theorem is as follows. The curves on S which are self-conjugate in the sense of G_2-differential geometry also have the property that their generalized tangent lines have second order contact with S, thus extending a familiar property of asymptotic lines.

H. Samelson (Ann Arbor, Mich.).

Kahane, J. P. Sur les propriétés des asymptotiques généralisées. Revue Sci. **85**, 286 (1947).

Given a 3-parameter family (II) of surfaces (which play the rôle of planes in Bouligand's differential geometry of the group G_2 [see the preceding review], assume that the family (Δ) of curves of intersection of the surfaces of (II) depends only on 4 parameters. The author shows by geometrical reasoning that the following two definitions of an asymptotic line C on a surface S are equivalent: (a) the member of (Δ) tangent to C has second order contact with S, (b) the curve C has second order contact with the member of (II) tangent to S, at the point in question. The two possible definitions of conjugate directions are also shown to be consistent, and the theorem of Koenigs is shown to hold in the G_2-differential geometry. *H. Samelson* (Ann Arbor, Mich.).

Finikoff, S. Certain periodic sequences of Laplace of period six in ordinary space. Duke Math. J. **14**, 807–835 (1947).

Let S and S' be two surfaces in one-to-one point correspondence, corresponding points being P, P'. It is well known that there is a net of curves on each surface whose tangents intersect the corresponding tangents of corresponding curves on the other surface. Such nets are called d-nets. The present paper discusses d-nets which are asymptotic on S and conjugate on S'. In section 1, among the theorems proved are the following. Given an arbitrary surface S_1 there is a family of surfaces S' depending on four arbitrary functions of one variable, each such surface S' having a conjugate net corresponding to the asymptotic net on S and corresponding tangents to these nets intersecting. To a given surface S corresponds a family of congruences Γ depending on two arbitrary functions of one variable. Each such congruence Γ is conjugate to the asymptotics of S_1 and there exists a family of transversal surfaces S' for each congruence Γ depending on two arbitrary functions of one variable, Γ being conjugate to a conjugate net on S'. Given an arbitrary surface S', there exists a family of conjugate d-nets depending on four arbitrary functions of one variable whose corresponding d-nets are asymptotic on S. If the points of intersection of corresponding tangents of two d-nets are the foci F_1, F_2 of the conjugate tangents $P'F_1$, $P'F_2$ of the d-net on S_1' then the lines PP', F_1F_2 are reciprocal lines with respect to the Darboux quadrics of S at P. The d-net on S' is harmonic. If the points $\cdots$, F_1^*, F_2, P', F_1, F_2^*, $\cdots$ describe the sequence of focal surfaces of the harmonic d-net of S', then the points F_1^*, F_1 are situated on one of the asymptotic tangents of S, and $F_2F_2^*$ are on the other. To each surface of this family there corresponds but one surface S', but there exists a family of surfaces S depending on four arbitrary functions of a variable such that to each surface S there correspond two surfaces S' and $S^{*\prime}$. The sequences of Laplace with respect to these latter two surfaces S', $S^{*\prime}$ coincide, are periodic and contain six distinct focal surfaces, S', S_1, S_2^*, $S^{*\prime}$, S_1^*, S_2 generated by the respective points P', F_1, F_2^*, $P^{*\prime}$, F_1^*, F_2. The asymptotics on each pair of opposite focal surfaces S' and S^*, S_1 and S_1^*, S_2 and S_2^* correspond while the congruences of lines $(F_1F_1^*)$, $(F_2F_2^*)$ are projectively applicable.

In the second section the defining system of differential equations of a general periodic sequence of Laplace of period six are developed, and the results are applied to the study of the special periodic sequences of Laplace considered in section 1. Among the general properties of the general sequence of period six may be listed the following. If two consecutive focal nets are harmonic, each focal net of the sequence is harmonic. If a focal net N_1 is harmonic, its opposite focal net N_4 is also harmonic; if three nets are harmonic all are harmonic. If a focal net N_1 has equal point invariants, its opposite net N_4 enjoys the same property. If the point invariants of three consecutive nets of the sequence are equal, all are equal. Finally a study is made of such periodic sequences whose focal net N_1 is harmonic with equal point invariants. *V. G. Grove* (Río Piedras, P. R.).

Charrueau, André. Sur des congruences de droites ou de courbes déduites d'une surface quelconque. C. R. Acad. Sci. Paris **225**, 792–794 (1947).

Soit σ une surface nondéveloppable, O_1, O_2, O_3 trois points différents et arbitraires et $\overrightarrow{OI_1}$, $\overrightarrow{OI_2}$ deux vecteurs quelconques. Soient enfin A un point de σ, α le plan tangent à σ en A, $\mathbf{n}$ un vecteur unitaire d'origine O et orthogonal à α, et $\overrightarrow{O_1M_1}=\overrightarrow{OI_1}\times\overrightarrow{O_1A}$, $\overrightarrow{O_2M_2}=\overrightarrow{OI_2}\times\overrightarrow{O_2A}$. Menons par M_1, M_2 des droites m_1 respectivement m_2 parallèles à $\mathbf{n}$. Lorsque A se change sur σ, m_1, m_2 engendrent deux congruences de droites. L'auteur étudie les propriétés des congruences (par exemple, les surfaces moyennes, les points focaux F_1, G_1 de m_1 respectivement F_2, G_2 de m_2, etc.).

La surface σ et le point A étant fixes et les vecteurs $\overrightarrow{OI_1}$, $\overrightarrow{OO_1}$, $\overrightarrow{OO_2}$, $\overrightarrow{OI_2}$ étant fonctions d'un même paramètre, les points M_1, M_2 et les points F_1, G_1, F_2, G_2 et aussi le point d'intersection $(F_1F_2, G_1G_2)\equiv P$ décrivent des courbes. L'auteur étudie les cas quand A se déplace sur σ ou A est fixe sur σ et quelques-uns des vecteurs $\overrightarrow{OI_1}$, $\overrightarrow{OI_2}$, $\overrightarrow{OO_1}$, $\overrightarrow{OO_2}$ sont invariables et quelques-uns ont des composantes (suivant trois axes fixes) égaux à des expressions linéaires du paramètre et trouve les différents covariants de la surface.

On peut étudier d'une manière analogue le cas d'une surface développable. [Voir aussi Bull. Sci. Math. (2) **70**, 127–148 (1946); C. R. Acad. Sci. Paris **225**, 620–622 (1947); ces Rev. **8**, 531; **9**, 158.] *F. Vyčichlo* (Prague).

Havlíček, K. Klein's representation of ruled surfaces. Acta Fac. Nat. Univ. Carol., Prague no. **172** (1939), 17–20 (1946).

First some of the author's work which points out that developable ruled surfaces correspond to minimal curves in the Klein representation for ruled surfaces is summarized again. [Cf. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **44**, 581–583 (1943); these Rev. **8**, 488.] Second, letting

E, F, G, L, M, N, K, H and P be the coefficients of the fundamental forms, the Gaussian curvature, the mean curvature and the parameter of distribution for a skew ruled surface, and k_1, k_2, k_3 the curvatures of the curve which corresponds to the ruled surface, the following theorems are stated. A representation with a given factor σ being given, it is possible to express the functions k_i by means of $E, F, \cdots, K, H, P$ and their derivatives. It is generally possible to choose a representation of a given ruled surface (i.e. to find the corresponding σ) such that an arbitrarily prescribed formula between the curvatures k_i (of a curve in V_4) and $E, F, \cdots, K, H, P$ of this given ruled surface holds.

A. Schwartz (State College, Pa.).

Santaló, L. A. Affine invariants of certain pairs of curves and surfaces. Duke Math. J. **14**, 559–574 (1947).

In this paper the author computes the simplest affine invariants for the following sets of curves and surfaces and gives their metric and affine characterizations: (1) two plane curves having a common tangent at two ordinary points: there are two nonprojective invariants; (2) two plane curves intersecting at an ordinary point: there are two nonprojective invariants; (3) two surfaces with common tangent planes at two ordinary points: there are three cases depending on whether the common tangent line is asymptotic on neither, one or both surfaces, and there are three, two and one invariants, respectively; only a particular combination of the first three invariants is projective; (4) two surfaces with common tangent line but with distinct tangent planes at two ordinary points: there are three cases again with two invariants whose product is projective in one case but with projective invariants only in the other cases.

A. Schwartz (State College, Pa.).

Tuganov, N. G. On basic lines on surfaces. Doklady Akad. Nauk SSSR (N.S.) **57**, 327–330 (1947). (Russian)

If a curve on a given surface is such that the asymptotic tangents of one family at all points of the curve intersect a given fixed plane in a straight line, the curve is called a basic curve of the surface. Although the author treats such lines intrinsically, it is evident that being basic depends on the fixed plane so that a curve may be basic with respect to one plane and not to another. The author takes the surface to be $z=f(x, y)$ and the fixed plane as $z=0$; this again is a severe specialization. In this case he finds the finite equations of the basic lines as the intersections of the surface with a cylindrical surface whose equation depends on two parameters. [If basic lines were defined intrinsically as curves whose asymptotic tangents intersect a straight line in space, such lines form a five parameter family.] An affine-basic line is a curve on the surface, the asymptotic tangents of one family at all its points being parallel to a plane associated with the curve. These affine-basic lines coincide with the basic lines if the fixed plane is taken as the plane at infinity. The finite equations of these lines also depend on two parameters. The author also considers the class of surfaces having conjugate nets of affine-basic lines and shows that this class depends on four arbitrary functions.

M. S. Knebelman (Pullman, Wash.).

Ancochea, Germán. On the duals of the theorems of Meusnier and Euler. Revista Acad. Ci. Madrid **41**, 189–195 (1947). (Spanish)

Meusnier's theorem refers to the plane sections of a two-dimensional surface whose planes pass through a fixed tangent line. Each plane cuts the surface in a plane curve, and the theorem states that the osculating circles to these curves at the point of contact of the tangent line all lie on a sphere. In the dual theorem we choose a range of points on a tangent line (which is not an asymptotic direction); with these as vertices construct the cones which are tangent to the surface, and consider the cones of revolution which osculate these along the given tangent line. The theorem states that these cones are all circumscribed to a sphere (of radius r). The author gives an original proof of this dual theorem based upon his definition of the "conical curvature" of a cone relative to a generator, a notion which is dual to the curvature of a plane curve at a point. The radius of conical curvature is shown to equal the tangent of the semi-vertex angle of the cone of revolution which osculates the cone along the given generator.

A new proof is also given of the dual to Euler's theorem, namely: if the chosen tangent makes an angle φ with the principal direction whose radius of curvature is r_1, then $r=r_1\cos^2\varphi+r_2\sin^2\varphi$, where r_1 and r_2 are the principal radii of curvature.

C. B. Allendoerfer (Haverford, Pa.).

Carbone, F. Sulle superficie di terzo ordine aventi un punto doppio biplanare o uniplanare. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 547–551 (1947).

The author studies the projective differential geometry of algebraic surfaces of the third degree which possess a double point O such that the tangent quadric cone with vertex at O is composed of two planes, distinct or coincident. If the two planes at O are distinct and if their line of intersection does not lie in the cubic surface, then the surface has two independent projective invariants. Moreover any two such cubic surfaces with the same projective invariants are projectively equivalent in four different ways. If the line of the two planes at O lies in the surface, then this cubic surface has only one projective invariant. Finally it is proved that a cubic surface which has a double point at O such that the tangent quadric cone with vertex at O consists of only one plane has no projective invariants, that is, any two such surfaces are projectively equivalent. The method is by reduction to canonical forms.

J. De Cicco (Chicago, Ill.).

Carbone, F. Sulle superficie analitiche aventi un punto doppio biplanare o uniplanare. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 551–554 (1947).

The author applies the results of the paper reviewed above to the study of the projective differential geometry of an analytic surface which possesses a double point O such that the tangent quadric cone with vertex at O is composed of two planes, distinct or coincident. The method is by appropriately approximating the given surface by a cubic surface with the same type of singularity at O. For all possible cases, the author reduces the equation of the surface to canonical form and establishes the existence of some of the projective invariants of the surface.

J. De Cicco.

Iwamoto, H. On Frenet's formulae. Tensor **7**, 44–49 (1944). (Japanese)

The geometrical interpretation of the Frenet-Serret formulae of m-dimensional surfaces in an n-dimensional Riemannian space was studied by many authors, e.g. Wong [Quart. J. Math., Oxford Ser. **11**, 146–160 (1940); these Rev. **2**, 20]. In this paper the author states the result of Wong in generalized form. First he discusses the angular relation between two linear vector subspaces in a Euclidean vector space and by use of his results he derives the geomet-

rical interpretation of Frenet's formulae for a subspace of a Riemannian space. *A. Kawaguchi* (Sapporo).

Urabe, M. On the proof of Rachevsky related to a subprojective Riemannian manifold. Tensor 4, 48–52 (1941). (Japanese)

In the book of Schouten and Struik [Einführung in die neueren Methoden der Differentialgeometrie, v. 2, 2d ed., Noordhoff, Groningen, 1938, p. 220] there is a proof of the theorem due to Rachevsky: a subprojective Riemannian space is conformally Euclidean when $n \geqq 3$ [Abh. Sem. Vektor- und Tensoranalysis 1, 126–142 (1933)]. This proof is made only for $n>3$ and not for $n=3$. This theorem is proved for $n=3$ in this paper. *A. Kawaguchi* (Sapporo).

Satoh, S. On local tetracyclic coordinates in a projective plane. Tensor 5, 87–88 (1942). (Japanese)

The theory of a family of conics in a projective plane was discussed in detail by A. Kawaguchi [Tôhoku Math. J. **28**, 126–146, 171–192, 202–211 (1927)]. This paper tries to discuss the theory from another point of view, that is, by considering a family of conics of which every member C has double contact with each conic of the given family; C is considered as an absolute in a locally non-Euclidean plane. Then the conic of the given family becomes a non-Euclidean circle in the locally non-Euclidean plane and the property of the family may be derived making use of tetracyclic coordinates by a method analogous to that used in Möbius differential geometry. *A. Kawaguchi* (Sapporo).

Tsuboko, M. On hypersurfaces with planes in E_4 as generating planes. Tensor 7, 24–34 (1944). (Japanese)

A one-parameter family of planes in a 4-dimensional projective space E_4 forms a hypersurface S which is called a ruled hypersurface. Then S must have a characteristic curve of which the tangent planes are generating planes of S. In this paper many properties of a ruled hypersurface are obtained relating to curves on S, e.g. characteristic curve, antiderivative curves and so on. The method employed is that of J. Kanitani. *A. Kawaguchi* (Sapporo).

Kimpara, M. On a mistake in Bompiani's theorem concerning the projective normal. Tensor 5, 47–50 (1942). (Japanese)

E. Bompiani has stated the following theorem [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 9, 39–44 (1929)] which is cited by Fubini and Čech [Introduction à la Géométrie Projective Différentielle des Surfaces, Paris, 1931, p. 94]. Let P_1 and P_2 be points of intersection of the asymptotic lines at two points P and P_0 on a curve C contained in a surface. The point P and the tangent at P_i $(i=1, 2)$ to the asymptotic line passing through P and P_i determine a plane π_i. The line of intersection of two planes π_1 and π_2 passes through P and generates a cone passing through the tangent at P to C, when the point P_0 varies on the curve C. The tangent plane to this cone Σ along the tangent at P to C always passes through the projective normal to the surface at P, even if we choose any curve passing through P as C. This paper shows that this theorem is not correct and that the tangent plane to the cone Σ along the tangent at P to C does not pass through the projective normal but generates a cone of third class which the author calls Bompiani's cone. Some properties of Bompiani's cone are given. *A. Kawaguchi* (Sapporo).

Kimpara, M. On a property of Green's edges in a space with a projective connection and generalization of Darboux's tangents. Tensor 4, 35–38 (1941). (Japanese)

J. Kanitani generalized Green's lines to a projectively connected 3-space [Proc. Phys.-Math. Soc. Japan (3) **22**, 343–358 (1940); these Rev. **2**, 21]. The author finds a new property of these generalized Green's lines and a new generalization of Darboux's tangents. Some properties of the latter are also stated. *A. Kawaguchi* (Sapporo).

Kimpara, M. Theory of surfaces in a three-dimensional space with a projective connection. Tensor 6, 33–44 (1943). (Japanese)

This paper is a sketch of the fundamental theory of surfaces in a three-dimensional space with a projective connection which was studied recently by J. Kanitani and M. Kimpara. The fundamental equations, line elements and normal edges of a surface are described. *A. Kawaguchi.*

Suguri, T. Some remarks on generalized projective transformations of affine connection. Tensor 5, 73–80 (1942). (Japanese)

In a previous paper [Mem. Fac. Sci. Kyūsyū Imp. Univ. (A) **2**, 91–106 (1941); these Rev. **3**, 311] the author has discussed projective transformations

$$\bar{\Gamma}_j^i(x, x^{(1)}) = \Gamma_j^i(x, x^{(1)}) + \varphi(x, x^{(1)})\delta_j^i + \psi_j(x, x^{(1)})x^{(1)i}$$

of generalized affine connections, in which the covariant differentiation is defined by $Dv^i/dt = dv^i/dt + \Gamma_j^i(x, x^{(1)})v^j$, where $x^{(1)i} = dx^i/dt$. In this paper the author obtains necessary and sufficient conditions for the functions φ and ψ_j in order that every one of the torsion tensors $S_{jk}^{\prime\prime i}$, S_{jk}^i and the curvature tensors $R_{hjk}^{\prime i}$, R_{hjk}^i remains unaltered by projective transformations of this connection. *A. Kawaguchi.*

Kanitani, J. Spaces with a linear connection reduced from a space with a projective connection. Tensor 4, 1–12 (1941). (Japanese)

The purpose of this paper is to find necessary and sufficient conditions for reducibility of various spaces with linear connections from a space with a projective connection Γ_{ij}^α $(\alpha = 0, 1, \cdots, n;\ i, j = 1, 2, \cdots, n)$. That is, a conformal connection is obtained from the projective one by introducing a metric and an affine connection by determination of a hyperplane at infinity. Furthermore, if both a hyperplane at infinity and a metric are given, a Euclidean connection appears whose special case is a Riemannian connection. The necessary and sufficient conditions are expressed by relations among the functions Γ_{ij}^α and their forms are somewhat complicated. *A. Kawaguchi* (Sapporo).

Muto, Y. On a slightly generalized conformal connection. Tensor 4, 69–71 (1941). (Japanese)

To define a conformal connection it is usually assumed that the parameters of connection Π_{Ak}^B depend on only the position x^i. This assumption is taken off in this paper and parameters are obtained which may depend not only on the position x^i but also on $x'^i, x''^i, \cdots$. As an example it is shown that the parameters of connection of an m-dimensional subspace depend on x^i as well as x'^i, when we consider as the tangent Möbius space the sphere M_m which has contact of second order with a curve lying on the subspace. *A. Kawaguchi* (Sapporo).

Muto, Y. Theory of subspaces in a space with a conformal connection. Tensor 5, 31–46 (1942). (Japanese)

A summary of the study of the theory of subspaces in a space with a conformal connection which has been done by S. Sasaki, K. Yano and Y. Muto. *A. Kawaguchi.*

Muto, Y. Curve length in a space with a conformal connection. Tensor 5, 60–65 (1942). (Japanese)

K. Yano and Y. Muto have defined the conformal arc length [Proc. Imp. Acad. Tokyo 17, 318–322 (1941); these Rev. 7, 332]. This paper shows that this conformal arc length is the same as Liebmann's parameter for a curve in a flat 3-space. Two geometrical interpretations of the conformal arc length are added. *A. Kawaguchi* (Sapporo).

Sasaki, S. Geometry of conformal connections. Tensor 4, 13–24 (1941). (Japanese)

The following problems are fully discussed. (1) Does conformal geometry in a Riemannian space contain the conformal geometry of Möbius as a special case? The answer is negative. (2) To find the relation between Cartan's theory of a space with conformal connection and the conformal geometry of the Princeton school. (3) Equivalence problem of spaces with normal conformal connections. (4) Conformal normal coordinate system and the theory of subspaces.
A. Kawaguchi (Sapporo).

Sasaki, S. On a relation between a Riemannian space which is conformal with Einstein spaces and normal conformally connected spaces whose groups of holonomy fix a point or a hypersphere. Tensor 5, 66–72 (1942). (Japanese)

Cf. the author's paper, Jap. J. Math. 18, 615–622 (1943); these Rev. 7, 330.

Sasaki, S., and Aoki, K. On a generalization of the theory of space form. Tensor 7, 35–43 (1944). (Japanese)

This paper is devoted to explaining in outline the relations between differential geometry and topology, chiefly restricted to the generalization of the theory of space form, especially the problem of metrization and continuation. It discusses the results obtained by H. Hopf, W. Rinow, S. B. Myers and É. Cartan on the relations between the differential geometries in the large and small. *A. Kawaguchi.*

Yano, K. On Sasaki's study of holonomy groups of spaces with normal conformal connections. Tensor 6, 68–85 (1943). (Japanese)

Cf. K. Yano, Proc. Imp. Acad. Tokyo 19, 444–453 (1943); these Rev. 7, 330.

Yano, K. The variable u^0 of T. Y. Thomas and the homogeneous coordinates of D. van Dantzig. Tensor 4, 38–48 (1941). (Japanese)

This paper makes clear the meaning of the zeroth variable u^0 of T. Y. Thomas and homogeneous coordinates of D. van Dantzig in a projectively connected space by interpreting them in a classical (flat) projective space by use of curvilinear coordinates and Cartan's "repère naturel." We can also see in the case of a classical projective space the relations among the various theories due to Cartan, Thomas, Veblen, van Dantzig, Schouten and Haantjes.
A. Kawaguchi (Sapporo).

Yano, K. On the fundamental theorem of conformal geometry. Tensor 5, 51–59 (1942). (Japanese)

By discussion of complete integrability conditions for the fundamental equations of a subspace in a conformally Euclidean space, this paper proves the following fundamental theorem. In order that a symmetric tensor g_{jk}, $n-m$ symmetric tensors M_{jkP} and $(n-m)(n-m-1)/2$ vectors L_{PQk} $(=-L_{QPk})$ for $j, k=1, 2, \cdots, m$; $P, Q=m+1, \cdots, n$, determine a V_m with $\rho^2 g_{jk}$, ρM_{jkP}, ρL_{PQk} as three kinds of conformal fundamental tensors, it is necessary and sufficient that these quantities satisfy not only three systems of equations which are analogous to those of Gauss, Codazzi and Ricci, but also other two systems of equations. The forms of the last two systems of equations are expressed explicitly. *A. Kawaguchi* (Sapporo).

Yano, K. Equations of paths in O. Veblen's projective space. Tensor 7, 65–72 (1944). (Japanese)

O. Veblen defined a projective connection by the functions $\Pi^\lambda_{\mu\nu}$ under three conditions $\Pi^\lambda_{\mu\nu}=\Pi^\lambda_{\nu\mu}$, $\Pi^\lambda_{0\nu}=\Pi^\lambda_{\nu 0}=\delta^\lambda_\nu$, $\Pi^\lambda_{\mu\nu,0}=0$ [J. London Math. Soc. 4, 140–160 (1929)]. In the representation of the space P_n with a projective connection into a space A_{n+1} with an affine connection, the geometrical interpretation of these three conditions was already discussed by J. H. C. Whitehead and K. Yano, and it is known that a geodesic in P_n is represented by a two-dimensional total-geodesic surface S_2 in A_{n+1}. This paper shows that S_2 can be expressed by any curve $x^\lambda(r)$ on S_2 in the form

$$\frac{d^2x^\lambda}{dr^2}+\Pi^\lambda_{\mu\nu}\frac{dx^\mu}{dr}\frac{dx^\nu}{dr}=\alpha\frac{dx^\lambda}{dr}+\beta\delta^\lambda_0$$

and inversely any solution of this equation represents an S_2. Moreover, the author discusses S_2, considering it as a subspace in A_{n+1}, and shows that its curvature tensor is identically equal to zero. Parameters of S_2 are also obtained for which Γ^a_{bc} for S_2 are all equal to zero. *A. Kawaguchi.*

Ohkubo, T. On a symmetric displacement in a Finsler space. Tensor 4, 53–55 (1941). (Japanese)

The parameters of connection Γ^{*i}_{jk} introduced by Cartan [Les espaces de Finsler, Actual. Sci. Ind., no. 79, Hermann, Paris, 1934] can be got from the postulates: (1) the covariant differentials are given by $\delta v^i=dv^i+\Gamma^i_{jk}v^jdx^k$, where Γ^i_{jk} are functions of x^i and x'^i, (2) Γ^i_{jk} are symmetric in j and k, (3) the connection is metric when the line element x'^i moves parallel to itself. The metric connection for a line element not moving parallel to itself is also got by Kawaguchi's method [Nederl. Akad. Wetensch., Proc. 40, 596–601 (1937)], that is, $\Lambda^i_j=\Gamma^i_{jk}dx^k+\frac{1}{2}g^{ik}\delta g_{jk}$; Λ^i_j is identical with the ω^i_j of Cartan. *A. Kawaguchi* (Sapporo).

Ohkubo, T. Descriptive geometry of paths. Tensor 5, 81–86 (1942). (Japanese)

In a previous paper [Tensor 1, 36–39 (1938)] the author studied the projective geometry of generalized paths in which any path is determined by three points and the differential equations of paths are given in the form $x'''^i+G^i(x, x', x'')=0$ which is invariant under any projective transformation $\bar{t}=(\alpha t+\beta)/(\gamma t+\delta)$ of the parameter t of the paths. This paper deals with the descriptive geometry of these generalized paths, that is, the invariant theory of the paths under any transformation of t. The equations of the paths become $x'''^i+G^i(x, x', x'')+G(x, x', x'')x'^i$ after a transformation of t, where the function G depends on the transformation but G^i does not. Hence one must discuss

the invariant theory of G^i under any transformation of coordinate system and $G^i \rightarrow G^i + Gx'^i$. The parameters of connection are obtained which correspond to Thomas's projective parameters and from Π^i_{jk} there are derived the parameters L^i_{jk} which are transformed by any transformation of coordinate system just as those of an affine connection. Other invariants are also calculated.

A. Kawaguchi (Sapporo).

Ohkubo, T. A generalization of Cartan's space. Tensor 6, 45–48 (1943). (Japanese)

Let us consider an $(n-1)$-ple integral

$$\int\int \cdots \int \{A_i^{\alpha\beta}(x^k, x^k_\sigma)x^i_{\alpha\beta} + B(x^k, x^k_\sigma)\}^{(n-1)/2} du^{\dot{1}} du^{\dot{2}} \cdots du^{\dot{n}-\dot{1}}$$

which is assumed to be invariant for any parameter transformation $\bar{u}^\alpha = \bar{u}^\alpha(u^\beta)$, where $x^i_\alpha = \partial x^i/\partial u^\alpha$, $x^i_{\alpha\beta} = \partial^2 x^i/\partial u^\alpha \partial u^\beta$ ($i, k = 1, 2, \cdots, n$; $\alpha, \beta = \dot{1}, \dot{2}, \cdots, \dot{n}-\dot{1}$); then we see that the functions $A_i^{\alpha\beta}$ must have the form $p_i E^{\alpha\beta}$, putting $p_i = (-1)^{i+1}\partial(x^1, \cdots, x^{i-1}, x^{i+1}, \cdots, x^n)/\partial(u^{\dot{1}}, \cdots, u^{\dot{n}-\dot{1}})$, and that $L = |E^{\alpha\beta}|^{1/(1-n)}$ is a function of x^i and p_i and homogeneous of dimension one with respect to p_i. It can be concluded that $g^{ij} = \frac{1}{2}\partial^2(L^2)/\partial p_i \partial p_j$ is a tensor; hence the author adopts it as fundamental tensor and determines from it the covariant differential of a vector $\delta v^i = dv^i + \Gamma^i_{jk} v^j dx^k + C^{i\alpha}_{jk} v^j dx^k_\alpha$ and that of a U-vector $\delta v^\alpha = dv^\alpha + \Pi^\alpha_{\beta k} v^\beta dx^k + P^{\alpha\gamma}_{\beta k} v^\beta dx^k_\gamma$ by use of Cartan's method for Finsler spaces. *A. Kawaguchi.*

Tonowoka, K. On a geometrical treatment of an $(n-1)$-ple integral of a certain kind. Tensor 7, 16–23 (1944). (Japanese)

This paper derives a symmetric tensor $g_{ij}(x^k, \partial x^k/\partial u^\alpha)$ from an integral

$$\int_{(n-1)} R(x^i, \partial x^i/\partial u^\alpha, \partial^2 x^i/\partial u^\alpha \partial u^\beta)^{1/p} du^{\dot{1}} \cdots du^{\dot{n}-\dot{1}}$$

which is assumed to be invariant for any transformation of parameters u^α ($\alpha, \beta = \dot{1}, \dot{2}, \cdots, \dot{n}-\dot{1}$), where R is a rational integral function of $\partial^2 x^i/\partial u^\alpha \partial u^\beta$. This is a generalization of the case studied by T. Ohkubo [see the preceding review] and the method employed is the same as that of Ohkubo.

A. Kawaguchi (Sapporo).

Tonowoka, K. On a metric displacement along a curve in a special Kawaguchi space. Tensor 4, 60–62 (1941). (Japanese)

A. Kawaguchi studied a space in which the arc length is defined by $s = \int\{A_i(x, x')x''^i + B(x, x')\}^{1/p} dt$ [Trans. Amer. Math. Soc. **44**, 153–167 (1938)]. The author starts in this paper from the differentiation (2.2) of this paper of Kawaguchi and defines a covariant derivative of the form $\mathfrak{D}v^k = dv^k/dt + (J^k_{\cdot ji}x'^j + I^k_{\cdot ji}x''^j)v^i$ along a curve, where line elements of second order are considered as space elements. This derivative $\mathfrak{D}v^i$ is an integral invariant for any transformations of the parameter t of the curve. It is shown also that there can be defined a tensor $g_{ij}(x, x', x'')$ which satisfies the relation $g_{ij}x'^i x'^j = F^2$, F being the integrand of the curve length. From this covariant derivative, which is not metric, the metric one follows by the method of Kawaguchi [Nederl. Akad. Wetensch., Proc. **40**, 569–601 (1937)]. The covariant differential as well as the base connections are defined too. *A. Kawaguchi* (Sapporo).

Michihiro, T. Theory of curves in a two-dimensional space with arc length $s = \int(A_i x''^i + B)^{1/p} dt$. Tensor 4, 63–66 (1941). (Japanese)

The theory of curves in the space described in the title is studied, making use of Kawaguchi's theory [Trans. Amer. Math. Soc. **44**, 153–167 (1938)]. The author finds the formulae which are analogous to Frenet's and Bouquet's formulae in the classical differential geometry. Necessary and sufficient conditions are stated for this space to reduce to an affine 2-space. *A. Kawaguchi* (Sapporo).

Bortolotti, Enea. Geometry of a system of partial differential equations. Tensor 4, 25–34 (1941). (Japanese)

[This is a translation of a paper which appeared in Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 323–337, Rome, 1942; these Rev. **8**, 405.] This paper deals with the invariant theory of a system of partial differential equations

$$\frac{\partial^2 x^i}{\partial u^\lambda \partial u^\mu} = H^i_{\lambda\mu}(u, x, \partial x/\partial u), \qquad i = 1, \cdots, n;\ \lambda, \mu = \dot{1}, \cdots, \dot{m},$$

under the transformation group $u^{\lambda'} = u^{\lambda'}(u)$, $x^{i'} = x^{i'}(x, u)$. Under consideration of the variables $y^\lambda = u^\lambda$, $y^i = x^i$ as y^A ($A = 1, \cdots, n, \dot{1}, \cdots, \dot{m}$) the parameters of an affine connection Γ^A_{BC} are determined from the equations of transformation of $H^i_{\lambda\mu}$ by putting some tensors equal to zero, but for projective transformations: ${}^*\Gamma^A_{BC} = \Gamma^A_{BC} + \delta^A_B \psi_C + \delta^A_C \psi_B$, ψ_C being an arbitrary covariant vector such that $\psi_i = 0$. The parameters Γ^A_{BC} are expressed by the functions $H^i_{\lambda\mu}$ and their first and second derivatives with respect to $p^i_\lambda = \partial x^i/\partial u^\lambda$. Inversely the functions $H^i_{\lambda\mu}$ can be expressed by Γ's. From these parameters Γ^A_{BC} we can get four kinds of covariant differentiations $\nabla_j \xi^\lambda$, $\nabla_j \xi^i$, $\nabla_\mu \xi^\lambda$, $\nabla_\mu \xi^i$ for two kinds of vectors ξ^λ, ξ^i. By these operations all invariants can be derived in a way analogous to that for a projective connection. *A. Kawaguchi* (Sapporo).

Kawaguchi, A. Views on higher order geometry of connections. IV. Tensor 4, 66–68 (1941). (Japanese)

[For part III cf. Tensor 3, 68–70 (1940); these Rev. **2**, 22.]

Miscellaneous remarks on recent studies of higher order geometry [Michihiro, Tonowoka, Bortolotti [cf. the three preceding reviews]; M. M. Johnson, Bull. Amer. Math. Soc. **46**, 269–271 (1940); H. V. Craig, ibid. **46**, 752 (1940); H. Hombu, Proc. Imp. Acad. Tokyo **16**, 90–96, 97–103 (1940); J. Fac. Sci. Kyûsyû Imp. Univ. A. **1**, 29–110 (1940); Kawaguchi and Hokari, Proc. Imp. Acad. Tokyo **16**, 313–325 (1940); Kawaguchi, J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **9**, 153–188 (1940); cf. these Rev. **1**, 273; **2**, 22, 23, 167; **3**, 20]. *A. Kawaguchi* (Sapporo).

Kawaguchi, A. Determination of the fundamental tensor in a five-dimensional space based on two-dimensional area. Tensor 6, 49–61 (1943). (Japanese)

The purpose of this paper is to determine the fundamental tensor g_{ij} from a given integral

$$O = \int\int \sqrt{2L(x^i, p_\alpha{}^i)}\, du^{\dot{1}} du^{\dot{2}}, \qquad i, j, \cdots = 1, 2, 3, 4, 5;\ \alpha, \beta = \dot{1}, \dot{2},$$

which is assumed to be invariant for any transformation of parameters u^α, and regarded as defining two-dimensional area in the space, where L is any given function which is

homogeneous of dimension two with respect to each of $p_1^i \equiv \partial x^i/\partial u^1$ and $p_2^i \equiv \partial x^i/\partial u^2$. For $p^{ij} = 2p_{[1}^i p_{2]}^j$ Plücker's identities $\Phi_h \equiv 3\epsilon_{ijklh} p^{ij} p^{kl} = 0$ ($\epsilon_{ijklh} = \epsilon_{[ijklh]} = \pm 1$ or 0) hold good and therefore $L_{ij} \equiv \partial L/\partial p^{ij}$ is indeterminate, but it is proved that the tensor $L'_{ij} = (2L)^{-1} L_{i\beta} L_{j\alpha} p^{\beta\alpha}$ is uniquely determined, where $L' = L + L^i \Phi_i$, $L^i = -(2L)^{-1} \epsilon^{ijhkl} L_{jh} L_{kl}$. For the same reason $L'_{ij,kl} \equiv \partial L'_{ij}/\partial p^{kl}$ is also indeterminate, but it is shown that the functions L^{ij} and ρ are determined uniquely from the relations $L^* = L' + L^{ij}\Phi_i\Phi_j$, $L'_{ij} = \rho g_{ik} g_{jl} p^{kl}$, where we put $g^{-1} g^{i_5 j_5} = (4! \times 4)^{-1} \epsilon^{i_1 i_2 \cdots i_5} \epsilon^{j_1 j_2 \cdots j_5} L^*_{[i_1 i_2 [j_1 j_2} L^*_{i_3 i_4] j_3 j_4]}$, g being the determinant $|g_{ij}|$. Then we see easily that the fundamental tensor g_{ij} is determined completely by the last expressions and we have

$$L^*_{ij,kl} = 2\rho g_{[i[k} g_{j]l]} + p^{rs} \frac{\partial}{\partial p^{kl}} (\rho g_{[i[r} g_{j]s]}).$$

In particular, for a Riemannian space in which $2L = g_{ik} g_{jl} p^{ij} p^{kl}$, the above relations become

$$L^* = \tfrac{1}{2} g_{ik} g_{jl} p^{ij} p^{kl}, \quad L^*_{ij,kl} = 2 g_{[i[k} g_{j]l]}, \qquad \rho = 1.$$

By use of g_{ij} we can define the connection in this space by the method of the author [J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. 9, 153–188 (1940); these Rev. 3, 20]. With a slight modification these considerations can be applied to a five-dimensional space based on three-dimensional area.

A. Kawaguchi (Sapporo).

Kawaguchi, Akitsugu. Die Differentialgeometrie höherer Ordnung. III. Erweiterte Parametertransformationen und P-Tensoren. J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. 10, 77–156 (1941).

[Bezüglich Teil I und II vgl. J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. 9, 1–152, 153–188 (1940); diese Rev. 2, 22; 3, 20.] In dem nach Verf. benannten Räumen, wird die Bogenlänge einer auf einen Parameter t bezogenen Kurve durch ein Integral der Form $\int f(x, x^{(1)}, \cdots, x^{(M)}) dt$ gemessen. Dabei bedeutet $x, x^{(1)}, \cdots, x^{(M)}$ ein Linienelement M-ter Ordnung. Die Untersuchungen, wann ein solcher Ausdruck eine Integralinvariante darstellt, führt zwangsläufig zu den Entwicklungen der vorliegenden Arbeit. In ihr wird ein N-dimensionaler Raum zugrunde gelegt, dessen Element nicht der Punkt, sondern das Linienelement M-ter Ordnung ist. Diese Elemente sind längs einer Kurve $x = x(t)$ gegeben. Zweck der vorliegenden Arbeit ist es nun den Einfluss von Transformationen des Parameters t systematisch zu untersuchen und damit diejenigen Eigenschaften, die sich auf die innere (intrinseke) Geometrie beziehen. Die Theorie der Koordinatentransformationen hat Verf. bereits in Teil I behandelt. Verf. definiert zunächst mit Hilfe eines sich bei Parametertransformationen ergebenden Gesetzes den Begriff des P-Tensors. Nachdem untersucht wird, wann eine Funktion Gruppen von Parametertransformationen oder solche schlechthin gestattet, können P-Skalardichten von vorgegebenem Gewicht definiert werden. Es zeigt sich dann insbesondere, dass das invariante Integral, welches dem Kawaguchi-Raume zugrunde liegt, zum Integranden eine P-Skalardichte vom Gewicht 0 besitzt. Aus diesen P-Skalardichten lassen sich kontra- und kovariante P-Vektoren bilden, die ihrerseits für die Bildung von Invarianten geeignet sind. Verf. führt auch einen kovarianten Ableitungsprozess ein, der sich im wesentlichen auf die oben erwähnten ko- und kontravarianten P-Vektoren stützt. Abschliessend gibt Verf. einen ausführlichen Literaturnachweis über die Arbeiten der letzten zehn Jahre.

O. Varga (Debrecen).

Kawaguchi, A. On various tensors appearing in the higher order geometry of connection. Tensor 6, 1–26 (1943). (Japanese)

In the higher order geometry it is necessary to use not only tensors in the ordinary sense but other kinds of tensors, that is, extensors introduced by H. V. Craig [Amer. J. Math. 59, 764–774 (1937)], weak or strong extensors and many-dimensional extensors introduced by A. Kawaguchi [Monatsh. Math. Phys. 48, 329–339 (1939); these Rev. 1, 87], P-tensors and U-tensors introduced by A. Kawaguchi [cf. the preceding review] and their generalizations. In this paper there are stated fundamental properties of these tensors and their applications to the geometry of higher order spaces.

A. Kawaguchi (Sapporo).

Kawaguchi, A. On certain metric spaces of higher order. Tensor 7, 73–77 (1944). (Japanese)

Consider an intrinsic Kawaguchi space of order m and of dimension n which is defined by $s = \int F(x, x^{(1)}, \cdots, x^{(m)}) dt$. This paper deals with the case where (1) $m \geqq n$ and (2) there are n linearly independent vectors in the system of m intrinsic Synge vectors $\mathfrak{E}_{\lambda i}$ ($\lambda = 1, \cdots, m$). Introducing the metric tensor $g_{ij} = \sum_a \mathfrak{E}_{ai} \mathfrak{E}_{aj} + \mathfrak{E}_{1i} \mathfrak{E}_{1j}$, there is considered an orthogonal ennuple $\mathfrak{E}_{ai}$ and the author defines a connection by $\sum \mathfrak{E}_{ai} d\mathfrak{E}_a^j = -\Gamma^j_{i[\lambda]k} dx^{[\lambda]k}$ ($x^{[\lambda]}$ means the λth derivative of x with respect to the curve length s). In this connection the orthogonal ennuples at any points are parallel to each other and the covariant differential of g_{ij} vanishes identically. It is also shown that the highest order of derivatives of x with respect to any parameter t which are contained in $\Gamma^j_{i[\lambda]k}$ is equal to exactly $2m-1$. The theory of P-tensors is frequently applied in this paper [cf. the second preceding review].

A. Kawaguchi (Sapporo).

Katsurada, Y. On the theory of curves in a higher order space with some special metrics. Tensor 7, 58–64 (1944). (Japanese)

The geometry in a space with the arc-length

$$s = \int \{A_i(x, x')x''^i + B(x, x')\}^{1/p} dt$$

was studied by A. Kawaguchi [Trans. Amer. Math. Soc. 44, 153–167 (1938)]. This paper is chiefly concerned with its special case: $A_i = a_{ik}(x)x'^k$, $a_{ik} = -a_{ki}$, $B = 0$, $n = 3$. First there is determined the transformation group for which the functional form of the integrand remains unaltered, and its geometrical interpretation is also considered. Then, defining the curvature and torsion, the author obtains the formulae which are analogous to Frenet's and Bouquet's formulae for a space curve. In conclusion there can be defined two kinds of geodesics: one is a parabolic cubic space curve which osculates the plane at infinity and the other is an ordinary parabola.

A. Kawaguchi (Sapporo).

Iwamoto, H. On the conformal theory of metric geometry of higher order. Tensor 7, 50–57 (1944). (Japanese)

Two Kawaguchi spaces of order M whose metrics are given by $F(x, \cdots, x^{(M)})$, $\bar{F}(x, \cdots, x^{(M)})$, respectively, are called mutually α-conformal if $\bar{F} = \rho(x, \cdots, x^{(\alpha)})F$ ($0 \leqq \alpha \leqq M-1$). In this paper the author derives many scalars and vectors which are invariant for the conformal transformation $\bar{F} = \rho F$. For instance he finds vectors $\Gamma_{i\mu}$ which are analogous to Synge vectors $E_{i\mu}$ and are transformed to $\bar{\Gamma}_{i\mu} = \rho \Gamma_{i\mu}$ by an $(M-1)$-conformal transformation. In conclusion the conformal covariant differential $\delta v^i = dv^i + \mathfrak{D}_1 \Gamma^i_j v^j$,

base connections $\delta x^{(\gamma)i}=(\delta^i_j-x^{(1)i}\Gamma_{\beta}\mathfrak{F}^{-1})dx^{(\gamma)j}+\sum_{\nu}\Lambda^{\nu i}_{\gamma j}dx^{(\nu)j}$ and conformal metric tensor $\mathfrak{G}_{ij}=\kappa^{-2}g_{ij}$ are determined. This paper seems to solve the fundamental problem of the conformal theory in Kawaguchi spaces completely.

A. Kawaguchi (Sapporo).

Mikami, M. Projective theory of a system of paths of higher order. Tensor 6, 86–94 (1943). (Japanese)

Outline of the study by H. Hombu and M. Mikami [Hombu, J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. 7, 35–94 (1938); Hombu and Mikami, Proc. Imp. Acad. Tokyo **18**, 595–601 (1942); Jap. J. Math. **17**, 307–335 (1941); these Rev. **7**, 334, 396].

Hokari, S. Geometry in an n-dimensional space based on the idea of K-dimensional volume. Tensor **4**, 72–77 (1941). (Japanese)

See the papers with the same title by A. Kawaguchi and S. Hokari [Proc. Imp. Acad. Tokyo **16**, 313–319, 320–325 (1940); these Rev. **2**, 167].

Hokari, S. On a geometrical treatment of a system of higher partial differential equations. Tensor **5**, 89–103 (1942). (Japanese)

In a previous paper [Proc. Imp. Acad. Tokyo **16**, 326–332 (1940); these Rev. **2**, 166] the author proved the theorem: If an intrinsic symmetric affinor $h^{(m)}_{\alpha\beta}$ $(\alpha, \beta=\dot{1}, \dot{2}, \cdots, \dot{K})$ with highest rank K exists, an intrinsic displacement $\nabla_\beta v^{i\alpha}=D_\beta v^{i\alpha}+\Gamma^i_{j\beta}v^{j\alpha}+G^\alpha_{\gamma\beta}v^{i\gamma}$ along an integral surface of a given system of partial differential equations $p^i_{\alpha_{(m)}}+H^i_{\alpha_{(m)}}(u^\beta, x^j, p^j_{\beta_{(1)}}, \cdots, p^j_{\beta_{(m-1)}})=0$ $(i, j=1, 2, \cdots, n)$ can be determined. The chief result of this paper is proof of the existence of such an affinor $h^{(m)}_{\alpha\beta}$. *A. Kawaguchi.*

Hashimoto, H. On the geometry of a system of partial differential equations of third order. Tensor **4**, 55–59 (1941). (Japanese)

This paper determines the intrinsic (i.e., invariant under transformations of the parameters u) connection as well as base connections from a system of partial differential equations of third order,

$$\frac{\partial^3 x^i}{\partial u^{\alpha_1}\partial u^{\alpha_2}\partial u^{\alpha_3}}+H^i_{\alpha_1\alpha_2\alpha_3}\left(u^\beta, x^j, \frac{\partial x^j}{\partial u^{\beta}}, \frac{\partial^2 x^j}{\partial u^{\beta_1}\partial u^{\beta_2}}\right)=0.$$

The author uses the idea of Kawaguchi and Hombu [J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **6**, 21–62 (1937)] and the elimination method for the equations of transformation of $H^i_{\alpha_1\alpha_2\alpha_3}$. *A. Kawaguchi* (Sapporo).

Bompiani, E. Enti geometrici definiti da sistemi differenziali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 887–894 (1946).

Maxia has shown [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 169–174 (1946); these Rev. **8**, 350] how a partial differential equation of second order can be made free from accidental representations by introducing geometric objects invariantly connected with the equation and interpreting the equation in a space with a certain linear connection dependent on these objects. In this paper the same is done for the equation of third order

$$\text{(I)} \qquad \partial_{hkl}z=\Gamma^{pq}_{hkl}\partial_{pq}z+\Gamma^{p}_{hkl}\partial_{p}z+\Gamma_{hkl}z.$$

It is proved that the Γ_{hkl} form an affinor and that the Γ^{pq}_{hkl} as well as the Γ^{pq}_{hkl} together with the Γ^{p}_{hkl} form a geometric object. If an arbitrary symmetrical linear connection Γ^p_{hk} is given, the Γ^{pq}_{hkl} and Γ^{p}_{hkl} can be expressed in terms of the Γ^p_{hk} and their first derivatives and two affinors $S_{hkl}^{\cdot\cdot\cdot pq}$ and $\Gamma_{hkl}^{\cdot\cdot\cdot p}$, the first symmetrical in hkl and pq and the second symmetrical in hkl. It is proved that there is one and only one connection Γ^p_{hk} such that

$$\Gamma^{pq}_{hkl}=3\Gamma^{(p}_{(hk}A^{q)}_{l)}+S_{hkl}^{\cdot\cdot\cdot pq},$$
$$\Gamma^{p}_{hkl}=\partial_{(l}\Gamma^{p}_{hk)}-2\Gamma^{\ \,t}_{(hk}\Gamma^{p}_{l)t}-S_{hkl}^{\cdot\cdot\cdot rs}\Gamma^p_{rs}+\Gamma_{hkl}^{\cdot\cdot\cdot p},$$
$$S_{hks}^{\cdot\cdot\cdot ps}=0.$$

By means of this connection the condition of integrability can be written in an invariant form.

If z, $\partial_h z$ and $\partial_{hk}z$ are considered as coordinates z^a in an E_N, $N=n(n+3)/2+1$, every solution of (I) can be interpreted as the parametric form of an X_n in this E_N. At every point of this X_n a pseudonormal can be constructed and all these pseudonormals pass through a fixed point. By means of this geometric interpretation it is possible to construct to every system of the form (I) an adjoint system, representing the same X_n in E_N in hyperplane coordinates. This relation is reciprocal and two adjoint systems have the same conditions of integrability. A system can only be self-adjoint if $\Gamma_{pqr}=0$. But in this case the geometric interpretation in E_N degenerates. *J. A. Schouten* (Epe).

Nakae, T. Das geometrische Objekt. Tensor **7**, 1–5 (1944). (Japanese)

Review of studies by A. Wundheiler [Abh. Sem. Vektor- und Tensoranalysis **4**, 366–375 (1937)], J. A. Schouten and J. Haantjes [Proc. London Math. Soc. (2) **42**, 356–376 (1937)] and S. Gołąb [Math. Ann. **116**, 768–780 (1939)].

Gołąb, St. Sur la théorie des objets géométriques. Ann. Soc. Polon. Math. **19** (1946), 7–35 (1947).

The author considers geometric objects with one component in a one-dimensional space X_1. Let the transformation that leads from the coordinate system ξ^1 to ξ^2 be written $\xi^2=\varphi(\xi^1)$ and the components of the object be denoted by Ω_1 and Ω_2. If the geometric object is of class n there exists an equation of the form

$$\Omega_2=f(\Omega_1, \xi^1, \varphi(\xi^1), \varphi'(\xi^1), \cdots, \varphi^{(n)}(\xi^1))$$

[cf. J. A. Schouten and J. Haantjes, Proc. London Math. Soc. (2) **42**, 356–376 (1937)]. If the function f does not depend explicitly on ξ^1 and $\varphi(\xi^1)$ the geometric object is called a pure differential object. The purpose of this paper is to find all the pure differential objects with one component of class $v=2$ and 3 [cf. G. Pensov, C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 563–566 (1946); these Rev. **9**, 67; Gołąb, Math. Z. **44**, 104–114 (1938)]. For $v=2$ the function $f(\Omega, \varphi', \varphi'')$ satisfies the functional equation

$$f(x, \beta_1\cdot\alpha_1, \beta_2\alpha_1{}^2+\beta_1\alpha_2)=f\{f(x, \alpha_1, \alpha_2), \beta_1, \beta_2\}$$

for $\alpha_1\neq 0$, $\beta_1\neq 0$ and $f(x, 1, 0)=x$. This equation leads to the following result. Every pure differential object of class two is a monotonic function of an object with the transformation $\Omega_2=\Omega_1/\varphi'-\varphi''/\varphi'^2$; Ω_1 is the parameter of an affine connection. In the same way it is proved that the geometric objects of class 3 are monotonic functions of objects with the transformation

$$\Omega_2=\Omega_1/(\varphi')^2-\tfrac{3}{2}(\varphi'')^2/(\varphi')^4+\varphi'''/(\varphi')^3.$$

In this case Ω_1 is a parameter of a projective connection.

J. Haantjes (Amsterdam).

Wang, Hsien-Chung. On Finsler spaces with completely integrable equations of Killing. J. London Math. Soc. **22**, 5–9 (1947).

The paper gives a solution of the problem of determining all Finsler spaces of $N>2$ dimensions which admit a

$\frac{1}{2}N(N+1)$ parameter group of motions. Such spaces are proved to be the Riemann spaces of constant curvature. There is some novelty in the method of proof in the sense that results on Lie groups and linear groups are used instead of the classical method of studying the complicated integrability conditions of the equations of Killing. *S. Chern.*

Berwald, L. Ueber Finslersche und Cartansche Geometrie. IV. Projektivkrümmung allgemeiner affiner Räume und Finslersche Räume skalarer Krümmung. Ann. of Math. (2) **48**, 755–781 (1947).

[Bezüglich Teil III vgl. Ann. of Math. (2) **42**, 84–112 (1941); diese Rev. **2**, 304.]

Verfasser definiert im Finslerschen Raume, indem durch $ds=L(x, dx)$ das Bogenelement definiert ist, ein Krümmungsmass R. Die Grössen der Finslerschen Geometrie sind bekanntlich Funktionen des Linienelementes $(x, \dot{x})$, das Krümmungsmass hingegen hängt ausserdem noch von einem willkürlichen von $\dot{x}$ linear unabhängigen Vektor η ab, ist also erst für eine 2-Stellung $(\dot{x}, \eta)$ definiert. Falls bei festem $(x, \dot{x})$ R unabhängig von η ist, so heisst der Raum ein Finslerscher Raum von skalarer Krümmung. Ist dieser Skalar eine Konstante, dann heisst der Raum ein Finslerscher Raum konstanter Krümmung. Im Finslerschen Raum sind die auf die Bogenlänge s bezogenen Extremalen Lösungen des Differentialgleichungssystems $d^2x^i/ds^2+2G^i(x, dx/ds)=0$. Die G^i sind eindeutig aus L bestimmbar und in den dx^i/ds positiv homogen von zweiter Dimension. Durch ein Differentialgleichungssystem vom obigen Typus in dem G^i der angegebenen Homogenitätsbedingung genügt, ist aber gerade ein allgemeiner affiner Raum bestimmt. An die Theorie dieser Räume schliesst sich die sogenannte projektive Theorie der bahntreuen Abbildungen an. Verfasser überträgt nun diese Theorie auf Finslersche Räume. Hauptzweck der Arbeit besteht nun darin, mittels Begriffsbildungen dieser Theorie gewisse wichtige Unterklassen von Finslerschen Räumen zu kennzeichnen. Es ergibt sich so als eines der Hauptergebnisse, dass die Finslerschen Räume von skalarer Krümmung eine Unterklasse der Finslerschen Räume mit der Projektivkrümmung Null sind. Verfasser gibt die Identitäten (Beziehungen zwischen Krümmungs und Torsionsgrössen) an, die die skalaren Räume unter denjenigen der Projektivkrümmung Null kennzeichnen. Es ergibt sich hieraus eine einfache Charakterisierung der Räume konstanter Krümmung. Betrachtet man in der Klasse der Finslerschen Räume von der Projektivkrümmung Null die Unterklasse der projektiv ebenen Räume, so ist diese gleichzeitig eine Unterklasse der Räume von skalarer Krümmung. Es sind diejenige Finslersche Räume, in denen sich in geeigneten Koordinaten die Extremalen durch $n-1$ lineare Gleichungen derselben darstellen lassen; sie bilden keine Unterklasse der Räume konstanter Krümmung. Schliesslich werden die Grundgleichungen für Finslersche Räume von konstanter Krümmung und geradlinigen Extremalen hergeleitet. Im einleitenden Teil der Arbeit, der sich auf allgemeine affine Räume und deren projektive Theorie bezieht, wird auch ein neues Ergebnis hergeleitet, indem Verfasser nachweist, dass das Verschwinden der Projektivkrümmung eines allgemeinen affinen Raumes notwendig und hinreichend dafür ist, dass derselbe bahntreu auf einen ebensolchen Raum der Affinkrümmung Null abgebildet werden kann.

O. Varga (Debrecen).

Kosambi, Damodar. Les invariants différentiels d'un tenseur covariant à deux indices. C. R. Acad. Sci. Paris **225**, 790–792 (1947).

It is well known that a general second order tensor g_{ij} may be decomposed into a symmetric part $g_{(ij)}$ and an antisymmetric part $g_{[ij]}$. With the aid of the tensor g_{ij} one may construct a Christoffel symbol of the first kind, Γ_{jki}. If g_{ij} is symmetric then the Γ_{jki} are the Christoffel symbols of the first kind for a Riemannian space. By use of a variational principle, the author shows that the antisymmetric symbols $\Gamma_{[jk]i}$, which involve only the derivatives of the antisymmetric tensor $g_{[ij]}$, are the components of a tensor. A necessary and sufficient condition for the vanishing of $\Gamma_{[jk]i}$ is stated, namely that g_{ij} be the curl of a vector. Further, by use of the cofactors of $g_{(ij)}$ (when this tensor is of rank n), the author introduces a general affine connection. The antisymmetric part of this connection is Cartan's torsion tensor. Some remarks are made as to the application of this idea to the general geometry of paths.

N. Coburn (Ann Arbor, Mich.).

Gonçalves Miranda, Manuel. Tensor calculus. Anais Fac. Ci. Pôrto **27**, 65–74, 129–160 (1942); **28**, 8–18 (1943). (Portuguese)

Notes on a course of lectures.

NUMERICAL AND GRAPHICAL METHODS

***Gloden, Albert. Table des Bicarrés N^4 pour $3001\leq N \leq 5000$.** A. Gloden, Luxembourg, 1947. 19 pp.

***Gloden, A., et Bonneau, J. Table des Bicarrés des Entiers de 5001–10000.** A. Gloden, Luxembourg, 1947. i+21 pp. 30 Belgian francs.

Barlow in his New Mathematical Tables [1814] gave the values of N^4 for $N=1(1)1000$, and Gloden listed the values of N^4, $N=1001(1)3000$, in 1946 [cf. these Rev. **8**, 171]. Hence there are now tables of N^4 for $N=1(1)10000$.

R. C. Archibald (Providence, R. I.).

***Five-Figure Tables of Natural Trigonometrical Functions.** Prepared by H. M. Nautical Almanac Office. London: His Majesty's Stationery Office; New York, British Information Services, 1947. iv+124 pp. 15/–, Great Britain; $4.00, U.S.A.

The main tables contain values of sine, tangent, cotangent and cosine at intervals of 10″; five decimals are given except for the cotangent from 0° to 26°, where five significant figures are given. An auxiliary table gives values of the cotangent at intervals of 1″ up to 7° 30′ to five significant figures. These appear to be the only published tables at such small intervals. *R. P. Boas, Jr.* (Providence, R. I.). five-place

***Palm, Conny. Table of the Erlang Loss Formula.** Tables of Telephone Traffic Formulae, no. I. C. E. Fritzes Hovbokhandel, Stockholm, 1947. ii+23 pp.

Putting $a_n=a^n/n!$, the tables give the values to six decimal places of the function $f_n(a)=a_n(a_0+\cdots+a_n)^{-1}$ for all integers $n\leqq 150$ in the range [0(.05) (.1) 20 (.05) 30 (1) 50, 52(4)100]. For large n and small a we have $f_n(a)<10^{-6}$ so that the number of actual entries decreases rapidly with increasing n. The formula gives the "proportion of lost calls" for the typical telephone exchange with n available circuits and the incoming traffic obeying a Poisson law with mean a. *W. Feller* (Ithaca, N. Y.).

✱Tables of the Bessel Functions of the First Kind of Orders Ten, Eleven, and Twelve. By the Staff of the Computation Laboratory. Harvard University Press, Cambridge, Mass., 1947. 644 pp. $10.00.

✱Tables of the Bessel Functions of the First Kind of Orders Thirteen, Fourteen, and Fifteen. By the Staff of the Computation Laboratory. Harvard University Press, Cambridge, Mass., 1947. 624 pp. $10.00.

[For reviews of previous volumes cf. these Rev. **8**, 406, 605.] These two volumes contain tables of $J_n(x)$ for $n=10$, 11, 12 and $n=13$, 14, 15, respectively. In both cases x runs from 0 to 25 at intervals of 0.001 and from 25 to 99.99 at intervals of 0.01.
A. Erdélyi (Pasadena, Calif.).

Neuschuler, L. On k-nomial tables for functions of three variables representable as a sum of products of functions of one variable. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 187–190 (1947).

In this paper the author considers the evaluation of functions of the form

$$u_n=f_n(x_1, x_2, x_3)=\Phi_n\left[\sum_{i=1}^{n}\{f_{i1}(x_1)f_{i2}(x_2)f_{i3}(x_3)\} + \phi_1(x_p)\phi_2(x_q)+\psi_1(x_q)\right]$$

with $p, q=1, 2, 3$, and $p\neq q$. His object is to reduce the problem to a form where only functions of two variables are used. Thus, for $n=1$, $p=1$, he writes

$$f_{12}(x_2)f_{13}(x_3)=\eta, \quad f_{11}(x_1)\eta+\phi_1(x_1)=F$$

and it follows that

$$u_n=\Phi(F, x_q), \quad F=F(\eta, x_1), \quad \eta=\eta(x_1, x_2).$$

For general n, the author shows that $2n+1$ steps are needed. The further application of ideas outlined in earlier papers [same C. R. (N.S.) **36**, 121–124 (1942); **43**, 142–146 (1944); these Rev. **4**, 202; **6**, 132] is also discussed.
J. C. P. Miller (London).

✱Burks, Arthur W., Goldstine, Herman H., and von Neumann, John. Preliminary Discussion of the Logical Design of an Electronic Computing Instrument. 2d ed. Institute for Advanced Study, Princeton, N. J., 1947. vi+42 pp.

This report gives a preliminary discussion of a high speed electronic digital computing machine. The machine is to make full use of the flexible and compact coding of problems which is possible when orders as well as numbers are stored in the high speed memory and can be operated on and modified according to the progress of the computation. The high speed memory is to be capable of storing 4096 groups of 40 binary digits each. This is to be done by 40 selectron tubes (now under development by the RCA laboratories at Princeton). This limited high speed memory is to be supplemented by a secondary storage on magnetic wires which are to be read into the selectrons in block at a rate of at least 25,000 pulses per second (already achieved experimentally). This, in turn, is to be supplemented by a dead storage in the form of a library of tapes containing tables, orders for routine types of problems or parts of problems, etc.

The input is to be by magnetic wires and a special typewriter with 16 symbols; the output by magnetic wires, a typewriter, and a fluorescent screen for viewing results. The code is to have between 32 and 64 orders, including $+$, $-$, $\times$, $\div$ (but not $\sqrt{\ }$). A tentative set of 21 orders are given which deal with the operations of programming internally, but do not include orders for handling input or output. They permit "double accuracy" computations.

The machine is to operate in the binary system. It is to be operated in parallel, using all 40 binary digits at the same time, rather than in sequence, as does the mercury tank EDVAC of the Moore School. When representing numbers, the 40 binary digits represent one algebraic sign and a 39 binary digit number, giving a precision of about $.9\times10^{-12}$. When representing orders, the 40 digits are broken into two orders of 20 each; each order has 12 binary digits of a storage address, 6 for an operation, and two left over. The code is thus a "one address code," and consequently, the control of the machine goes through the high speed memory in sequence, taking orders as they come, except when instructed to skip to a different location.

The authors do not plan to have a "floating binary point," but believe they can either allow for it by suitable choices of scale factors as is done on a differential analyser, or program for it. They remark: "Furthermore, there is considerable redundancy in a floating binary point type of notation, for each number carries with it a scale factor, while generally speaking a single scale factor will suffice for a possibly extensive set of numbers. By means of the operations already described in the report a floating binary point can be programmed. While additional memory capacity is needed for this, it is probably less than that required by a built-in floating binary point since a different scale factor does not need to be remembered for each number."

With regard to checking: "One method of checking which is under consideration is that of having two identical computers which operate in parallel and automatically compare each other's results." It is also planned to be able to operate the machine step-by-step so that troubles can be localized. The report also contains a detailed description of how the authors plan to perform the basic operations of $+$, $-$, $\times$, $\div$, as well as a careful study of the roundoff rules.
R. W. Hamming (Murray Hill, N. J.).

✱Goldstine, Herman H., and von Neumann, John. Planning and Coding of Problems for an Electronic Computing Instrument. Institute for Advanced Study, Princeton, N. J., 1947. ii+69 pp.

This report discusses the general philosophy of coding problems for a large scale digital computing machine, and applies it to a number of relatively simple examples, using the orders introduced in the report reviewed above. The authors first observe that the coding of almost all interesting mathematical problems is not static in the sense that the orders are given in a fixed sequence which is to be traversed once by the control of the machine, but rather it is dynamic in the sense that the control of the machine goes repeatedly through some sets of storage registers and that each time it finds that some of the orders have been changed. These changes may be of a predictable type as one would find in an induction process, or they may depend on the numbers to be computed as in a method of successive approximation where the number of iterations is not known in advance. In order to grasp this dynamic situation in its entirety, the authors propose the use of a "flow diagram" which symbolically represents the path the control of the machine is to follow as it goes about carrying out the instructions it finds. Simple and multiple inductions are indicated by one or more closed oriented loops. Each such loop has an alternative box which decides when to go around again and when to go elsewhere. Further boxes are added to the flow dia-

gram to indicate the logical situation at selected points. These flow diagrams emphasize the logical aspects of the problem and subordinate the purely arithmetical aspects. After completion of the flow diagram, the actual coding is fairly easy, though tedious.

The report gives a number of examples, showing details, of the coding of simple problems. These problems include coding a floating binary point for division, square rooting using an iterative system, conversion to and from the binary system and "double accuracy" arithmetical operations. It also includes a rule of thumb for estimating times of operations in microseconds. *R. W. Hamming.*

Lyndon, Roger C. The Zuse computer. Math. Tables and Other Aids to Computation **2**, 355–359 (1947).

The author gives a brief account of a German relay-type digital computer. The instrument is apparently not completely built, but work is going forward. As planned it will have a memory capable of storing 1024 numbers, each of five decimal digits. The input and output seem to be somewhat crude; e.g., the only form of output is a bank of lamps to indicate a result in decimal form. The multiplication time is of the order of one second, which is roughly comparable with existing nonelectronic computers. It uses punched film to carry its program instructions; the control is of a somewhat primitive nature since there is no provision for handling directly inductive processes nor is there means for effecting a dichotomy. In a footnote the author states that the inventor, K. Zuse, now plans to remove these defects from the machine's control organ. *H. H. Goldstine.*

Bloh, Z. Š. On the theory of the λ-shaped mechanism of Čebyšev. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1944**, 559–565 (1944). (Russian)

Bloh, Z. Š. On the most advantageous dimensions for mechanisms of the Čebyšev-Evans type. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1944**, 836–840 (1944). (Russian)

In the first paper the Čebyšev and Evans mechanisms, each composed of three movable links, are compared with regard to (1) the length of the approximate straight line drawn, (2) the magnitude of the deviation from a straight line, (3) the deviation from uniformity of motion relative to uniform rotation of one crank. If b is the length of the base, and the crank turns through an angle of one radian, the three items enumerated are for Č and E, respectively: (1) $0.65b$ and $0.3b$, (2) $0.0015b$ and $0.052b$, (3) 1% and 10%.

In the second paper, a selected compromise design is investigated for deviations from linearity and uniformity for different portions of the trajectory. *M. Goldberg.*

Dubois, Fr. Die Schöpfungen Jakob und Alfred Amsler's auf dem Gebiete der mathematischen Instrumente anhand der Ausstellung in Museum Allerheiligen systematisch dargestellt. Mitt. Naturforsch. Ges. Schaffhausen **19**, 209–273 (1 plate) (1944).

This is a catalog of a museum collection of some of the mathematical instruments (or their photographs) designed and built by the Amsler firm in Switzerland since 1852. The basic theory, line sketches, diagrams and photographs are shown. The instruments discussed include the following.

(1) The familiar polar planimeter. Some of its variant forms are a single-cell pantograph combination, a lazy-tongs pantograph attachment for measuring fur and leather hides and the special disc rolling surface to reduce the errors due to the irregularities of paper surfaces. It has been built into several industrial measuring devices like the one in which the velocity of steam flow measured by a floating valve is integrated to give the total flow, and another in which the area of a motion-pressure steam diagram is measured to give the work per cycle. (2) The linear planimeter. These have been made to measure moments (up to the fourth) as well as the area. (3) The Durand-Amsler radial planimeter for measuring the area of the equivalent rectangular plot directly from the polar plot. A variation of this planimeter has been made for the integration of the product of two independent variables. (4) The stereographic planimeter for measuring the true area from a stereographic plot. (5) The ball integrator and the ball differentiator. These have been used in dynamometers which make records of motion, velocity and acceleration. (6) A cone differentiator which matches the measured velocity by a calibrated velocity from a rotating cone. (7) A curvimeter in which a cross-line on a magnifying lens is kept tangent to a small contour as it is traced. A pantograph enlarges the motion enabling the roller to measure the length of the curve. (8) A hydraulic device which measures and records the flow of water through a Venturi gauge. *M. Goldberg.*

Salzer, Herbert E. Table of coefficients for repeated integration with differences. Philos. Mag. (7) **38**, 331–338 (1947).

For the k-fold integral

$$J=\int_{x_0}^{x_1}\cdots\int_{x_0}^{x}\int_{x_0}^{x}f(x)(dx)^k,$$

where $x_1-x_0=h$, the tabular interval, and $x=x_0+ph$, the author gives two formulae:

$$h^{-k}J=f(x_0)/k!+\sum_{n=1}^{m}G_n^{(k)}\Delta^n f(x_0)+R_m,$$

using advancing differences,

$$h^{-k}J=f(x_0)/k!+\sum_{n=1}^{m}H_n^{(k)}\nabla^n f(x_0)+R_m',$$

using backward differences, where

$$G_n^{(k)}=\frac{1}{n!}\int_0^1\cdots\int_0^p\int_0^p p(p-1)\cdots(p-n+1)(dp)^k,$$

$$H_n^{(k)}=\frac{1}{n!}\int_0^1\cdots\int_0^p\int_0^p p(p+1)\cdots(p+n-1)(dp)^k.$$

Recursion formulae are given for $G^{(k)}$ and $H^{(k)}$, such as the exact values for $G^{(2)}$ and $H^{(2)}$ expressed by the values of Milne-Thomson's Bernoulli polynomials $B_m^{(n)}(x)$ for $x=1$. A table of numerical values of G and H at 8–12 decimal places concludes the paper. *E. Bodewig* (The Hague).

Lemaître, G. Calcul des intégrales elliptiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 200–211 (1947).

W. Bartky a donné un procédé de calcul de l'intégrale complète de troisième espèce qui étend à ces intégrales le procédé de Gauss pour le calcul des intégrales de première et de deuxième espèces basé sur l'emploi de la transformation de Landen et de la moyenne arithmético-géométrique [Rev. Modern Physics **10**, 264–269 (1938)]. L'auteur montre que les formules de Bartky sont applicables au calcul de l'intégrale incomplète de troisième espèce. Il considère

l'intégrale de troisième espèce sous la forme

$$\int_0^{\varphi_0} \frac{m_0 a_0 \cos^2\theta + n_0 r_0 b_0 \sin^2\theta}{a_0\cos^2\theta + r_0 b_0 \sin^2\theta} \cdot \frac{d\theta}{\sqrt{a_0^2\cos^2\theta + b_0^2\sin^2\theta}} = \int_0^{\varphi_0} U_0 \frac{d\theta}{\Delta_0}.$$

En effectuant une série de transformations de Landen, il trouve

$$\int_0^{\varphi_0} U_0 \frac{d\theta}{\Delta_0} = 2^{-k} \int_0^{\varphi_k} U_k \frac{d\theta}{\Delta_k} + \sum_{l=0}^{k-1} \frac{m_l - n_l}{2^{l+2}} B_{l+1},$$

où $a_{l+1} = \frac{1}{2}(a_l + b_l)$, $b_{l+1} = (a_l b_l)^{\frac{1}{2}}$ (moyenne a–g);

$$\sin(2\varphi_l - \varphi_{l+1}) = (c_{l+1}/a_{l+1}) \sin\varphi_{l+1},$$

respectivement $\tan(\varphi_{l+1} - \varphi_l) = (b_l/a_l)\tan\varphi_l$, $c_{l+1} = \frac{1}{2}(a_l - b_l)$ (transformation de Landen);

$$r_{l+1} = \left(\frac{1+r_l}{2}\right)^2 \frac{1}{r_l} \cdot \frac{b_{l+1}}{a_{l+1}} = \left(\frac{1+r_l}{2}\right)^2 \frac{1}{r_l} \cdot \frac{2(a_l b_l)^{\frac{1}{2}}}{a_l + b_l};$$

$$m_{l+1} - n_{l+1} = \frac{m_l - n_l}{2} \cdot \frac{r_l - 1}{r_l + 1}; \quad m_{l+1} = \frac{m_l + n_l}{2} = m_l - \frac{m_l - n_l}{2};$$

$$\begin{aligned} B_l = \int_0^{\varphi_l} \frac{\cos\theta\, d\theta}{a_l \cos^2\theta + r_l b_l \sin^2\theta} &= [a_l(r_l b_l - a_l)]^{-\frac{1}{2}} \tan^{-1}\left\{\frac{r_l b_l - a_l}{a_l}\right\}^{\frac{1}{2}} \cdot \sin\varphi_l, && rb > a, \\ &= a_l^{-1}\sin\varphi_l, && rb = a, \\ &= [a_l(a_l - r_l b_l)]^{-\frac{1}{2}} \\ &\quad \times \log\tan\left[\tfrac{1}{4}\pi + \tfrac{1}{2}\sin^{-1}\left\{\frac{a_l - r_l b_l}{a_l}\right\} \cdot \sin\varphi_l\right], && rb < a. \end{aligned}$$

S. C. van Veen (Delft).

Herzberger, M., and Morris, R. H. A contribution to the method of least squares. Quart. Appl. Math. 5, 354–357 (1947).

The authors develop a special method for solving a system of normal equations with a very small determinant. The opinion of the reviewer, however, is that the method involves more laborious calculations than the Gaussian algorithm. *E. Bodewig* (The Hague).

Many, Abraham, and Meiboom, Saul. An electrical network for determining the eigenvalues and eigenvectors of a real symmetric matrix. Rev. Sci. Instruments 18, 831–836 (1947).

A new electrical network is described for finding the eigenvalues and eigenvectors of secular equations. The real symmetric matrix of order n is represented by n LC-circuits, each one coupled to each of the others by two equal condensers. In contrast to more conventional circuits, the roots are found by varying the frequency of excitation, instead of the magnitude of the units themselves. *G. Kron.*

Kincaid, W. M. Numerical methods for finding characteristic roots and vectors of matrices. Quart. Appl. Math. 5, 320–345 (1947).

The matrix is supposed to have linear elementary divisors. The paper deals with the usual methods, for instance, Aitken's, and with new devices. Among the latter may be mentioned the use of matrix polynomials. Let λ_i' be approximate values of the roots λ_i of the matrix A. Then the matrix $P_i(A)$, where

$$P_i(\lambda) = (\lambda - \lambda_1') \cdots (\lambda - \lambda_n')/(\lambda - \lambda_i'),$$

has all roots near zero except $P_i(\lambda_i)$. So the usual iteration process applied to $P_i(A)V$, where V is arbitrary, will converge rapidly. The degree of the polynomial can be diminished by choosing V orthogonal to the characteristic column vector or row vector X_p or Y_p, respectively, belonging to λ_p, $p \neq i$. At the end of the paper the author gives a specialization of the above general theory for solving an algebraic equation by means of the characteristic vectors $X_i = (1, \lambda_i^{-1}, \lambda_i^{-2}, \cdots, \lambda_i^{-n+1})$ of the companion matrix (Bernoulli's method). *E. Bodewig* (The Hague).

Morris, J. The escalator process for the solution of damped Lagrangian frequency equations. Philos. Mag. (7) 38, 275–287 (1947).

The author extends the escalator process for the solution of Lagrangian frequency equations to similar equations having a damping term. The modes of such equations have orthogonal properties analogous to those for ordinary Lagrangian frequency equations. For example, let the two dual systems be: $f_{1i}(\lambda)x + f_{2i}(\lambda)y + f_{3i}(\lambda)z = 0$ and $f_{i1}(\lambda)x + f_{i2}(\lambda)y + f_{i3}(\lambda)z = 0$, where $i = 1, 2, 3$ and $f_{rs}(\lambda) = a_{rs} - b_{rs}\lambda - c_{rs}\lambda^2$. Using the principle of the escalator process the author determines first the roots μ and modes (x_k, y_k) of the system $f_{1i}(\mu)x + f_{2i}(\mu)y = 0$, $i = 1, 2$, and similarly (x_k', y_k') of the dual system. Here the arbitrary proportionality factor of the modes is chosen in a certain manner. Then the roots λ of the original system are determined from $\sum_{r=1}^4 P_r P_r'/(\mu_r - \lambda) - f_{33}(\lambda) = 0$, where $P_r = f_{13}(\lambda)x_r + f_{23}(\lambda)y_r$, $P_r' = f_{31}(\lambda)x_r' + f_{32}(\lambda)y_r'$. The modes follow from $x/z = -\sum P_r' x_r/(\mu_r - \lambda)$, $y/z = -\sum P_r' y_r/(\mu_r - \lambda)$, $x'/z' = -\sum P_r x_r'/(\mu_r - \lambda)$, $y'/z' = -\sum P_r y_r'/(\mu_r - \lambda)$.

E. Bodewig (The Hague).

Gutenmaher, L. I. Electrical multidimensional models with amplifiers. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1947**, 511–528 (1947). (Russian)

From the bibliography appended, it is clear that this paper is one of a considerable series on the general topic of continuous-variable ("analogy") electrical computing devices. The present article is principally concerned with solving (a) algebraic equations with complex coefficients, (b) pairs of linear partial differential equations with variable coefficients, (c) the Maxwell equations for three- or two-dimensional regions. For problem (a) the main point consists in means for obtaining voltages that are 90° out of phase and then in dealing with linear combinations of such voltages. In problem (b) the partial derivatives with respect to x are replaced by difference quotients, while time-derivatives are retained as such. This technique is extended to the Maxwell equations of part (c). *H. Wallman.*

Gradštein, I. S. The solution of systems of linear equations by L. I. Gutenmaher's electrical models. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1947**, 529–584 (1947). (Russian)

Systems of linear differential equations with constant coefficients are prepared for solution by means of electronic circuits by converting them into systems of simultaneous equations of first order, and then using a matrix scheme based on use of the Laplace transform. There is an extended study of the mathematical questions of stability and convergence of the feedback systems involved in this "electronization," and a certain amount of engineering data.

H. Wallman (Cambridge, Mass.).

Korol'kov, N. V. The results of the development and testing of an experimental apparatus for the solution of systems of differential equations. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1947**, 585–596 (1947). (Russian)

On the basis of previous theoretical work, chiefly by Gutenmaher, a small electronic differential analyzer for ordinary differential equations with constant coefficients has been built. Graphical solutions are presented on an oscilloscope. Considerable flexibility of equation set-up exists and much of standard differential-analyzer technique is available. An example is furnished by the expedient of replacing the second-order equation $\ddot{u}_1+\omega^2u_1=kt$ (initial conditions $u_1(0)=\dot{u}_1(0)=0$) by the set of simultaneous first-order equations $\dot{u}_1=u_2$, $\dot{u}_2+\omega^2u_1-u_3=0$, $\dot{u}_3-u_4=0$, $\dot{u}_4=0$ (initial conditions $u_1(0)=u_2(0)=u_3(0)=0$, $u_4(0)=k$); the purpose of the last two equations, in u_3 and u_4, is of course to generate $u_3=kt$. Oscillograms are given for the solution of this problem, as of others that are more complicated. The accuracy is as good as 2% in simple cases. *H. Wallman.*

Kind-Schaad, G. Lösung von Eigenwertproblemen mittels Lochkartenmaschinen. Schweiz. Arch. Angew. Wiss. Tech. **13**, 161–168 (1947).

The author describes the use of punched cards for the solution of characteristic roots and vectors by the method of iteration. *E. Bodewig* (The Hague).

Diaz, J. B., and Weinstein, Alexander. Schwarz' inequality and the methods of Rayleigh-Ritz and Trefftz. J. Math. Phys. Mass. Inst. Tech. **26**, 133–136 (1947).

The authors' aim in this note is to establish both the Rayleigh-Ritz and Trefftz procedure, in the case of quadratic functionals, by a simple and direct application of the Schwarz inequality and the Green formula. The work of this paper appears to be a simpler approach than the one given by Friedrichs [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1929**, 13–20]. *A. Gelbart* (Princeton, N. J.).

Isaacs, Rufus. Numerical determination by use of special computational devices of an integral operator in the theory of compressible fluids. II. Determination of the coefficients of the integral operator by interpolatory means. J. Math. Phys. Mass. Inst. Tech. **26**, 165–181 (1947).

For part I cf. Bergman and Greenstone, same vol., 1–9 (1947); these Rev. **8**, 541.

Watson, E. J. Formulae for the computation of the functions employed for calculating the velocity distribution about a given aerofoil. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2176**(8661), 8 pp. (1945).

In order to determine the velocity distributions about an arbitrary aerofoil, it is necessary to evaluate the functions ψ', ϵ and ϵ' when ψ is given numerically. If the values ψ_m of ψ are specified at $2N$ points equally spaced about the circle into which the aerofoil is transformed, the application of Poisson's integral is not easy and it is more convenient to follow Germain's procedure: to approximate ψ by a trigonometric polynomial which coincides with ψ_m at the given points and to assume that $\epsilon_m=\sum\alpha_p\psi_{m+p}$, $\epsilon_m'=\sum\beta_p\psi_{m+p}$, and so on. A table gives the coefficients $\alpha_p, \cdots$ for ϵ_m, ϵ_m', ψ_m', $\int\psi_m$, $\int\epsilon_m$ and explicit formulae. *E. Bodewig.*

Engelfriet, J. Une théorie générale de récurrence en matière d'assurance sur la vie et contre l'invalidité. Verzekerings-Arch. **27**, 1–78 (1947).

ASTRONOMY

Buchanan, Daniel. A transformation to the normal form. Rend. Circ. Mat. Palermo **62**, 385–387 (1941).

The equations of variation at the Lagrangian libration points of the restricted problem of three bodies are transformed into normal form in case the mass ratio μ of the finite masses is chosen so that the indicial equation has $\pm i$ as double roots. *M. H. Martin* (College Park, Md.).

Egerváry, E. On a generalization of the Lagrangean solution of the problem of three bodies. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 793–795 (1947).

The author discusses the planar motion of three particles of equal mass subject to forces derived from a potential dependent only on the principal moments of inertia. By introduction of special Lagrangian coordinates, it is shown that the general motion can be described as a composition of two equilateral triangle motions. It is shown that the equations can be completely integrated in special cases which include in particular the Lagrangian triangle solution and a case analyzed by Sokolov [same C. R. (N.S.) **46**, 95–98 (1945); these Rev. **7**, 224]. *W. Kaplan.*

Lahaye, Edmond. Les chocs binaires simultanés réels dans le problème des N corps. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 678–693 (1947).

Lahaye, Edmond. Les chocs triples réels dans le problème des N corps. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 89–104 (1947).

The author determines series expansions in fractional powers of the time for the coordinates of the N bodies in the case of several simultaneous binary collisions and in the case of a triple collision. [Cf. A. Wintner, The Analytical Foundations of Celestial Mechanics, Princeton University Press, 1941, pp. 324–338; these Rev. **3**, 215].

W. Kaplan (Ann Arbor, Mich.).

de Jekhowsky, Benjamin. Simplification du système principal d'équations des méthodes d'Olbers et de Gauss pour la détermination des orbites paraboliques. C. R. Acad. Sci. Paris **225**, 489–491 (1947).

The methods of Olbers and of Gauss and Gauss-Merton for the determination of a parabolic orbit from three observations lead to a system of five equations from which the geocentric distances at the first and third dates are determined. It is shown that, if the interval of time between the extreme dates is sufficiently small, the system may be simplified and reduced to two equations, analogous to those arising in the method of Lagrange-Andoyer. Hence the method of solution for these equations proposed by the author [same C. R. **222**, 783–785 (1946)] may be employed. An example in which the interval is 9 days is given as an illustration. *D. Brouwer* (New Haven, Conn.).

Chazy, Jean. Sur les satellites artificiels de la Terre. C. R. Acad. Sci. Paris **225**, 469–471 (1947).

The author considers the motion of a rocket which has been propelled through dense portions of the terrestrial atmosphere up to an altitude r, where the propelling force stops and the missile moves under the influence of gravity

alone. Assuming an air resistance at this altitude to be proportional to the velocity of the missile, the author proves that the rocket can indeed become a satellite of the earth provided that its velocity v at r happens to be between the limits $(gr)^{\frac{1}{2}} \leqq v < (2gr)^{\frac{1}{2}}$, where g stands for the gravitational acceleration at r and the motion is in a direction sensibly normal to the radius-vector. *Z. Kopal.*

Esclangon, Ernest. Sur la transformation, en satellites permanents de la Terre, de mobiles issus de la surface du globe. C. R. Acad. Sci. Paris **225**, 513–515 (1947).

The author shows that a rocket launched from the surface of the earth can become its permanent satellite if the propulsion can be maintained beyond the altitude at which the air resistance becomes insignificant. He points out that the splinters of a projectile exploded at great heights could become such satellites. *Z. Kopal* (Cambridge, Mass.).

ten Bruggencate, P. Die Rotation der Milchstrasse und die Theorie der Schnelläufer. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1947**, 4–11 (1947).

In Oort's theory of galactic rotation it is assumed that our sun and the great majority of the nearby stars move in very nearly circular orbits. The author assumes the circular velocity of rotation at the sun to be of the order of 300 km/sec. The marked asymmetry in the motions of the high velocity stars is then generally explained through the escaping from our galaxy of stars with velocities (with reference to the galactic center) in excess of 365 km/sec. The author shows that it is difficult to reconcile the value 1.22 for the ratio between the velocity of escape and the circular velocity with the model of a highly flattened galaxy with a marked central condensation; this is the model usually said to be indicated by the known dimensions of our galaxy and the observed values of the Oort constants A and B.

The author favors a reduction in the assumed value for the circular velocity as a possible way out of the dilemma. As an alternative he suggests a prevalence of noncircular orbits for the vicinity of the sun. The reviewer wishes to add that approximately circular motion with a velocity of the order of 300 km/sec can be reconciled with the generally accepted model of our galaxy, provided we assume that the velocity of 365 km/sec sets an effective upper limit to the radius of our galaxy, rather than that it indicates an escape limit. *B. J. Bok* (Cambridge, Mass.).

Krall, G. Sulla formazione delle galassie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 221–228 (1947).

This paper consists of several remarks supplementary to a paper of Armellini on galactic structure [same Rend. (8) **1**, 486–493 (1946); these Rev. **8**, 495]. It is shown that the model proposed is consistent with the principle of "adiabatic invariance," and that the stable motion of the system forms a stationary solution of the basic dynamical equations. *W. Kaplan* (Ann Arbor, Mich.).

Zagar, Francesco. Sui problemi fondamentali della cosmografia planetaria. Rend. Sess. Accad. Sci. Ist. Bologna (N.S.) **45**, 16–27 (1941).

This paper is a criticism of the hypothesis, advanced by Armellini [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1**, 121–126 (1940); these Rev. **8**, 410] and others, that the Newtonian gravitational force should be supplemented by a term proportional to dr/dt, r being the mutual distance. It is indicated that the proposed correction leads to values of constants of the solar system in sharp disagreement with the observed values. *W. Kaplan.*

Milne, E. A. The equation to the arm of a spiral nebula. Astrophys. J. **106**, 137–142 (1947).

The paper is a continuation of the author's investigations of the dynamics of spiral nebulae on his kinematic relativity and the associated theory of gravitation (in which the ordinary constant of gravitation is a linear function of the fundamental t-measure of time) [cf. Monthly Not. Roy. Astr. Soc. **106**, 180–199 (1946); these Rev. **8**, 607]. The equation which should represent the arm of a spiral nebula (i.e., the locus of the present positions of a set of revolving particles) on the basis of Milne's kinematic relativity is expressed in various alternative forms, the simplest of which appears to be the parametric representation

$$\theta-\theta_1=(\gamma_1 M t_1^2/r_1^3)^{\frac{1}{2}} \log t/t_1,$$

where r, θ are current plane polar coordinates; γ, the (variable) constant of gravitation; M, the mass of the nucleus of the nebula; and t, the fundamental time in terms of which the nebulae are mutually receding; t_1 denotes an arbitrary epoch at which the position of a particle was r_1, θ_1, and at which $\gamma=\gamma_1$. It is shown that the number of orbital revolutions completed by a particle up to the present moment is equal to the net number of convolutions of the arm between $\theta=\theta_1$ and the present position of the particle. Thus, as we pass from the inner parts of a spiral to the outer parts, the number of complete orbital revolutions first increases to a certain limit and then decreases again, ultimately approaching zero. Milne's theory also shows that, in reality, we must expect the region of emission of the particles constituting spiral arms to be concentrated in a very small range of θ in the vicinity of some fixed azimuth θ_1. This appears to be in agreement with certain earlier conclusions of Jeans and Lindblad. *Z. Kopal.*

Auluck, F. C., and Kothari, D. S. A note on the minimum radius for degenerate stellar masses. Philos. Mag. (7) **38**, 368–370 (1947).

In an earlier paper with the same title [same Mag. (7) **35**, 783–786 (1944); these Rev. **6**, 244] the nonrelativistic virial theorem $2K+W=0$ (K = kinetic energy, W = potential energy) was used in the relativistic case. Here the relativistic virial theorem is derived. For inverse-square forces between particles it gives, for gas at pressure p in a volume V, $3pV=W+\sum mv^2(1-v^2/c^2)^{-\frac{1}{2}}$ and for a stellar mass in gravitational equilibrium

$$3\int_0^r p\,dV-\int_0^r GM(r)dM(r)/r=3p(r)V.$$

T. G. Cowling (Bangor).

RELATIVITY

Kaufmann, W. Zur Mechanik hoher Geschwindigkeiten. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1946**, 26–28 (1946).

The author shows how the addition theorem for relative velocities in special relativity theory may be derived from the theory of mechanics if it be postulated that $E=mc^2$ and that the law of conservation of momentum of a particle should be invariant in form under space-time transformations. *H. C. Corben* (Pittsburgh, Pa.).

Jordan, P. Zur projektiven Relativitätstheorie. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1945**, 39–41 (1945).

The group G of transformations under which the field equations are invariant is generated from proper transformations and from coordinate transformations. The group G is isomorphic with the group H_5 of homogeneous transformations in five variables $X_0, \cdots, X_4$. If Z is the central of H_5, and N a certain self-conjugate subgroup, then G is simply isomorphic with H_5/Z and N/Z is simply isomorphic with the group of proper transformations in four dimensions. There exists also a subgroup K of H_5 such that K/Z is simply isomorphic with the group of coordinate transformations in four dimensions. It is argued that it should therefore be possible to express the field equations in a very simple form in terms of the five variables $X_0, \cdots, X_4$. *H. C. Corben* (Pittsburgh, Pa.).

Gregory, Christopher. Non-linear invariants and the problem of motion. Physical Rev. (2) **72**, 72–75 (1947).

In this paper the usual relativistic field equations are modified by the addition of nonlinear invariants. The Einstein, Infeld and Hoffman approximation method is applied to the modified field equation and also to the equations of motion. The approximation is carried only as far as the Newtonian approximation. *M. Wyman.*

Walker, A. G. The invariants of kinematical relativity. Philos. Mag. (7) **38**, 316–324 (1947).

The author defines "strict tensors" in connection with a preassigned symmetry group. This concept is applied to Milne's kinematical relativity and to the author's generalization of Milne's theory. *A. Schild* (Princeton, N. J.).

Datta Majumdar, Sudhansu. A class of exact solutions of Einstein's field equations. Physical Rev. (2) **72**, 390–398 (1947).

For line elements of the form $ds^2=g_{ab}dx^adx^b+g_{44}dt^2$ it is shown that the assumption that g_{44} is a function of the electrostatic potential φ implies $g_{44}=A+B\varphi+\frac{1}{4}\varphi^2$, where A, B are arbitrary constants. For the particular case $g_{44}=\frac{1}{4}(\varphi+c)^2$ it is shown that the line element must have the form $ds^2=e^wdt^2-e^{-w}((dx^1)^2+(dx^2)^2+(dx^3)^2)$. By a suitable substitution the field equations reduce to Laplace's equations. *M. Wyman* (Edmonton, Alta.).

Karmarkar, K. R. An important particular case of the problem of equivalence. Bull. Calcutta Math. Soc. **39**, 30–32 (1947).

This paper gives three scalars whose invariance is necessary for the equivalence of two spherically symmetrical line elements

(1) $ds^2=-Adr^2+2Cdrdt+Ddt^2-B(d\theta^2+\sin^2\theta d\phi^2)$,

where A, B, C, D and the coordinate transformation are functions of r, t only and $\partial B/\partial r\neq 0$. One of the scalars is B. The other two are expressed algebraically in terms of A, B, C, D and their partial derivatives of order not exceeding one and two, respectively. Element (1) always admits the special orthogonal form

$$ds^2=-e^\lambda d\rho^2+e^\nu d\tau^2-\rho^2(d\theta^2+\sin^2\theta d\phi^2).$$

The scalars are then $B=\rho^2$, e^λ and $\partial\nu/\partial\rho$. When two of the scalars are independent, they in a sense determine the transformation, and substitution will decide the question of equivalence. The case where no two are independent is discussed along the same lines. *J. M. Thomas.*

Karmarkar, K. R. A new theorem on the transformability of a line-element into the spherically symmetric form. Proc. Indian Acad. Sci., Sect. A. **26**, 52–55 (1947).

The author shows that a necessary and sufficient condition for a line element of the form

$$ds^2=-A(dx^1)^2-B(dx^2)^2-C(dx^3)^2+Ddt^2$$

to be the transform of a spherically symmetric line element is that the geodesic equations admit an integral of the type $ax^1+bx^2+cx^3=0$, where a, b, c are constant. *M. Wyman* (Edmonton, Alta.).

Vescan, Théophile-T. Note sur une nouvelle solution des équations de la gravifique relativiste et ses conséquences cosmologiques. C. R. Acad. Sci. Paris **225**, 278–280 (1947).

For a Friedmann type universe the total mass M is taken in the form $M=M_0+M_1e^{-\epsilon t}$. Cosmological consequences of the solution resulting from the above assumption are examined. *M. Wyman* (Edmonton, Alta.).

Yang, C. N. On quantized space-time. Physical Rev. (2) **72**, 874 (1947).

Snyder's theory of quantized space-time [same Rev. (2) **71**, 38–41 (1947); **72**, 68–71 (1947); these Rev. **8**, 412, 608] invariant under Lorentz transformations, is not invariant under translations unless the space-time coordinates form a continuum, which is inconsistent with the existence of a minimum length. Other properties of space-time, e.g., homogeneity, would require invariance under a wider group of transformations. Two examples are given. First, the group of linear transformations leaving the left-hand side of (1) $-x_0^2+x_1^2+x_2^2+x_3^2+\xi^2=R^2$ invariant is approximated to by the product of the Lorentz group and the group of translations for points on the de Sitter pseudosphere (1) for which $|x_\mu|\ll R$ ($\mu=0, 1, 2, 3$). The infinitesimal elements of these transformations give the angular- and linear-momentum operators L_i, M_i, p_μ ($i=1, 2, 3$). Their commutators are, e.g., (2) $[p_i, p_j]=i\hbar L_k/R^2$, $[x_i, p_j]=i\hbar\xi\delta_{ij}/R$. Second, x_μ, ξ are taken as Hermitian operators which, after a Lorentz transformation in five dimensions, are restored to their original forms by a unitary transformation: (2) are still valid. Commutators $[x_\mu, x_\nu]$, $[\xi, x_\mu]$ are not determined by invariance but are postulated in a form proportional to a^2, where a is a length. The fifteen operators $L_i/\hbar$, $M_i/\hbar$, $Rp_\mu/\hbar$, x_μ/a, ξ/a satisfy the same commutation relations as the infinitesimal elements of the group of Lorentz transformations in six dimensions with the basic form $-\eta_0^2+\eta_1^2+\eta_2^2+\eta_3^2+\eta^2+\zeta^2$. A possible solution for these operators gives discrete eigenvalues for the space coordinates. *C. Strachan* (Aberdeen).

Kunii, S. The problem of two bodies in general relativity. Tensor 5, 1–14 (1942). (Japanese)

There are three methods of discussing the problem of two bodies. The first is based on static solutions of the field equations [Bach, Weyl and Silberstein], the second is to solve the field equations or equations of geodesics for motion of a material point approximately [Droste, de Sitter, Levi-Civita, Eddington and Clark]. The last is to discuss the motion of celestial bodies from the field equations in the exterior regular region by the method of successive approximations, where a celestial body is regarded as a singular point in the gravitational field [Einstein, Infeld, Hoffmann, Robertson]. This paper explains these methods systematically and states some opinions of the author.

A. Kawaguchi (Sapporo).

Takeno, H. On spaces in cosmology. Tensor 5, 15–30 (1942). (Japanese)

The aims of this paper are the following: (1) to define a Riemannian space V_4 on which the cosmology will be based, independently of any coordinate system; (2) to find the fundamental form ds^2 in any coordinate system which may have physical meaning; (3) to find the actual form of transformation between these coordinate systems; (4) to find the group of motions in V_4.

A. Kawaguchi (Sapporo).

Sibata, T. Cosmology in terms of wave geometry and the theory of nebulae and the solar system. Tensor 6, 62–67 (1943). (Japanese)

Outline of the theory developed by Y. Mimura, T. Iwatsuki, T. Sibata, M. Takeno, K. Itimaru, K. Sakuma, etc. [J. Sci. Hirosima Univ. Ser. A. **8** (1938); **11** (1942); cf. these Rev. **3**, 63].

A. Kawaguchi (Sapporo).

Beck, Guido. Sur la possibilité d'une cinématique générale. Anais Fac. Ci. Pôrto **28**, 65–72 (1943).

The paper also appeared as Centro de Estudos de Mat. Fac. Ci. Pôrto. Publ. no. **5** (1943); these Rev. **4**, 286.

MECHANICS

Pailloux, Henri. Mouvements à deux paramètres: extension de la formule de Savary. C. R. Acad. Sci. Paris **225**, 662–664 (1947).

Un trièdre de sommet A, défini par des vecteurs unitaires $\mathbf{i}, \mathbf{j}, \mathbf{k}$ rectangulaires, caractérise un solide S_2 en mouvement par rapport à un solide S_1, sa position dépendant de deux paramètres α, β. Pour un point P on a $\mathbf{P}=\mathbf{A}+x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, $\mathbf{dP}=\mathbf{dA}+\mathbf{d\omega}\times\mathbf{AP}$, $\mathbf{dA}=\mathbf{V}_1 d\alpha+\mathbf{V}_2 d\beta$, $\mathbf{d\omega}=\mathbf{\Omega}_1 d\alpha+\mathbf{\Omega}_2 dp$. Le déplacement de la position α, β à la position $\alpha+d\alpha$, $\beta+d\beta$ est une rotation, pourvu que $Hd\alpha^2+2Kd\alpha d\beta+Ld\beta^2=0$, $H=\mathbf{V}_1\cdot\mathbf{\Omega}_1$, etc. Il existe donc deux rotations possibles, autour d'axes D et D'. Soit PTT' la droite passant par P, et s'appuyant sur D et D'. Alors: $\mathbf{dP}=\boldsymbol{\delta\omega}\times\mathbf{TP}+\boldsymbol{\delta'\omega}\times\mathbf{T'P}$. Pour les points P_1, P_2, lieux dans S_1, S_2 du point coincidant P, on a $\mathbf{dP}_2-\mathbf{dP}_1=\boldsymbol{\delta\omega}\times\mathbf{TP}+\boldsymbol{\delta'\omega}\times\mathbf{T'P}$. L'auteur donne des formules condensées pour l'indicatrice d'une surface Σ_1 enveloppe de Σ_2 entrainée par S_2, en utilisant quelques résultats d'une note précédente [C. R. Acad. Sci. Paris **224**, 1539–1540 (1947); ces Rev. **8**, 608].

O. Bottema.

Ingram, W. H. The modal oscillations of discrete dynamical systems. Philos. Mag. (7) **38**, 51–61 (1947).

The author studies some of the algebraic questions which arise when one replaces the matrix integral-differential equations of a one-dimensional continuous system by sums. The continuous case was studied by the author previously [Bull. Amer. Math. Soc. **48**, 153–162 (1942); these Rev. **3**, 242].

A. E. Heins (Pittsburgh, Pa.).

DeCicco, John. Constrained motion upon a surface under a generalized field of force. Bull. Amer. Math. Soc. **53**, 993–1001 (1947).

In the author's terminology a generalized field of force is a field of force such that the force acting upon a particle is determined by the position of the particle and by the direction (without regard to sense) in which the particle is moving. This concept seems to be entirely ad hoc, and one which does not correspond to any physical reality. The first part of the paper is devoted to the derivation of a large number of formulae relating to the motion of a particle which is constrained to move on a smooth surface and is subjected to a generalized field of force. The author does not use the approach by way of Lagrange's equations, and consequently this part of the paper is more complicated than is apparently necessary.

Let u and v denote the coordinates on the surface, and let $\varphi(u, v, v')$ and $\psi(u, v, v')$ denote the components of the generalized force. (Here $v'=dv/du$.) Then in general the differential equation $\psi-v'\varphi=0$ defines ∞^1 curves, which are called lines of force. The latter part of the paper is devoted mainly to proving the following theorem. If the particle starts from rest at a point O on the surface, the ratio of the geodesic curvature of the trajectory at O to the geodesic curvature at O of the line of force passing through that point is $(1-\lambda)/(3-\lambda)$, where $\lambda=(\psi_{v'}-v'\varphi_{v'})/\varphi$. This is a generalization of a well-known theorem of Kasner, who dealt with the case in which $\varphi_{v'}=\psi_{v'}=0$.

L. A. MacColl (New York, N. Y.).

Tatevsky, V. On some forms of equations of dynamics and their applications. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **17**, 520–529 (1947). (Russian. English summary)

In addition to the two customary choices of variables used to characterize a dynamical system, namely (1) the coordinates and their time derivatives and (2) the coordinates and the moments, the author considers two more choices, namely (3) the moments and their time derivatives and (4) the time derivatives of both the coordinates and the moments. He introduces in addition to the Lagrangian and the Hamiltonian two new characteristic functions, M and K, corresponding to the third and fourth choices of variables. He studies systematically the equations of motion and the variational principles for all these choices of variables. It is noted that in some cases a system that is nonholonomic for one choice of variables may reduce for another choice to one that is holonomic. In conclusion the author treats two applications; in both instances he emphasizes that the comparative advantages (especially for approximate calculations) of the different choices of variables depend on the peculiar properties of the system treated. In the case of small vibrations studied by the author before [same Zhurnal **15**, 445–458 (1945); these Rev. **8**, 101], he obtains, in addition to equations previously used by others, one that seems

new. The anharmonic oscillator is considered, and treated in part graphically for the cases when the potential energy is a polynomial in the p and in the q spaces.
G. Y. Rainich (Ann Arbor, Mich.).

de Franchis, Franco. Una condizione sufficiente per l'esistenza dell'integrale scalare generalizzato delle aree con riferimento a particolari problemi dinamici. Rend. Circ. Mat. Palermo 62, 377–381 (1941).

Hydrodynamics, Aerodynamics

Rivlin, R. S. Hydrodynamics of non-Newtonian fluids. Nature **160**, 611 (1947).

The author considers an incompressible fluid with stress, rate of strain relations

$$t_{xx}=2\theta a+2\psi A'+p, \cdots,$$
$$t_{yz}=\theta f+\psi F', \cdots,$$

where

$$a=\partial u/\partial x, \cdots, \quad f=\partial v/\partial z+\partial w/\partial y, \cdots,$$
$$A'=a^2+\tfrac{1}{4}(g^2+h^2), \cdots, \quad F'=\tfrac{1}{2}gh+(b+c)f, \cdots,$$

etc., p is the hydrostatic pressure; θ, ψ are arbitrary functions of the two strain velocity invariants. For a Newtonian fluid, $\psi=0$ and θ is the viscosity. The author describes two motions of the fluid for which he states normal surface forces proportional to ψ (where ψ is constant) must exist. Explicit expressions for the normal forces are given but the nature of the fluids which permit of such motions is not discussed. *N. Coburn* (Ann Arbor, Mich.).

Pyškin, B. A. Doubly spiral motion of a fluid in a straight duct of semicircular profile. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1947**, 1015–1019 (1947). (Russian)

Une masse de fluide parfait, incompressible, enfermée dans un cylindre indéfini à section droite semi-circulaire, est soumise à l'action des forces dérivant d'un potentiel; le mouvement de la masse: (1°) est permanent, (2°) est symétrique par rapport au plan de symétrie du tube, (3°) ne dépend pas de l'abcisse x de la section droite. Soient x, r, φ les coordonnées cylindro-polaires d'un point courant, l'origine étant prise sur l'axe de révolution du tube. L'auteur se propose l'étude du régime dans deux cas: (a) les lignes de courant coïncident avec les lignes de tourbillons; le rapport rot $\mathbf{V}/\mathbf{V}=k$, k étant un scalaire constant dans la section droite du tube; (b) rot $\mathbf{V}$ est parallèle à Ox; les composantes radiale et tangentielle de $\mathbf{V}$ étant respectivement égales à $-(\lambda/r)\partial\xi/\partial\varphi$ et $\lambda\partial\xi/\partial r$ (où ξ désigne la composante de $\frac{1}{2}$ rot $\mathbf{V}$ suivant Ox et λ un scalaire qu'on suppose, à priori, constant dans toute la section droite). En admettant, de plus, que les composantes de la vitesse soient de la forme $F(\varphi)R(r)$, l'auteur trouve la solution des deux problèmes sous forme finie et pousse la discussion jusqu'aux applications numériques. *J. Kravtchenko* (Grenoble).

Shiffman, Max, and Spencer, D. C. The flow of an ideal incompressible fluid about a lens. Quart. Appl. Math. **5**, 270–288 (1947).

The flow is the continuous irrotational flow about a solid formed of a pair of intersecting spheres in uniform motion along its axis in a fluid at rest at infinity. The potential function of this flow is discussed by "analytic continuation." It is in general infinitely many valued, new values being reached along paths linked with the circle of intersection of the two spheres and is considered in terms of toroidal coordinates. The theory of dipoles in the associated "Riemann space" is sketched and a principle of inversion in spherical surfaces (through the given circle of intersection) is established. Following this inversion principle the required potential can be expressed as an infinite series and this expression is transformed into contour and real integrals. The symmetrical solid and the case where the angle between the spheres is a rational multiple of π are given special consideration and a comparison is made with the classical solution for nonintersecting spheres. Numerical results for the "virtual mass" (and associated ones relating to the impact of a sphere on a plane fluid surface) in the symmetrical case are expressed graphically.
A. J. Macintyre (Aberdeen).

Pretsch, J. Die Stabilität einer ebenen Laminarströmung bei Druckgefälle und Druckanstieg. Jahrbuch **1941** der Deutschen Luftfahrtforschung, I58–175 (1941).

The author investigates the stability of a laminar boundary layer with a pressure rise or a pressure drop. He uses the Hartree profiles as typical velocity distributions. For cases of decreasing pressure, a binomial approximation is used; for cases of increasing pressure, a sinusoidal approximation is used. The result is then applied to general pressure distributions through the use of the Pohlhausen parameter. Applying the Kármán-Pohlhausen method for the calculation of the boundary thickness, the stability limit for any boundary layer may be calculated provided the pressure distribution is known. These calculations of stability limit were carried out for seven Kármán-Trefftz profiles at zero lift, the NACA 2409 airfoil at various lift coefficients, and the sinusoidal wall.

The results can only be interpreted as giving qualitative indications of the Reynolds number of transition, as the stability limit is usually much lower than the transition Reynolds number. The author proposes to investigate the transition number by calculating the actual amount of amplification [see the following review]. *C. C. Lin.*

Pretsch, J. Die Anfachung instabiler Störungen in einer laminaren Reibungsschicht. Jahrbuch **1942** der Deutschen Luftfahrtforschung, I54–I71 (1942).

The author continues the investigation of the paper reviewed above by calculating the amplification of small disturbances superposed on Hartree velocity distributions. It is found that the amplification in the narrow instability region in the case of decreasing pressure is much smaller compared with that in the wide instability region in the case of rising pressure. This explains the fact that the transition point for large Reynolds numbers usually occurs after the pressure minimum. *C. C. Lin* (Cambridge, Mass.).

Lamla, Ernst. Eine einfache Näherungsbetrachtung zur Umströmung schlanker Profile und Rotationskörper im unterkritischen Gebiet. Jahrbuch **1940** der Deutschen Luftfahrtforschung, I66–171 (1940).

Werden sehr schlanke Profile oder Rotationskörper in ihrer Längsrichtung wirbelfrei und reibungslos umströmt, so vereinfachen sich die Gleichungen für das Potential φ und die Stromfunktion ψ in erster Näherung sehr weitgehend. Es wird darauf hingewiesen, dass die beiden so gewonnenen Gleichungen für φ und für ψ bei strenger Orthogonalität der Potential- und der Stromlinien im all-

gemeinen nicht zugleich erfüllt werden können. Sodann wird gezeigt, dass die Näherung zu den verschiedenen Fassungen der Prandtlschen Regel führt, und dass man mit Hilfe der Ergebnisse der Theorie der inkompressibeln Flüssigkeiten fast ohne jede Rechnung Näherungslösungen angeben kann, und zwar zahlenmässig am Beispiel des elliptischen Zylinders und des Joukowsky-Profils im freien Raum und am Kanal. Am Schluss wird die Erreichung höherer Näherungen diskutiert. *Author's summary.*

Schmieden, C. Die konforme Abbildung von Tragflügelprofilen mit Krümmungssingularitäten. Jahrbuch **1942** der Deutschen Luftfahrtforschung, I106–I110 (1942).

Es ist bekannt, dass Profile mit Krümmungssprüngen in der Umgebung der Nase in der Druckverteilung an dieser Stelle eine senkrechte Tangente besitzen. In der vorliegenden Arbeit wird an einer Reihe von Profilen die Auswirkung eines solchen Krümmungssprunges auf die weitere Umgebung dieser Stelle in der Druckverteilung untersucht, ebenso auch die Auswirkung der nächst höheren Singularität eines Krümmungsknickes, d.h. eines Sprungen in der Ableitung der Krümmung nach der Bogenlänge. Darüber hinaus gibt das entwickelte Abbildungsverfahren die Möglichkeit, allgemein Profile mit gegebenem Krümmungsverlauf konform abzubilden. *Author's summary.*

Greenberg, J. Mayo. Some considerations on an airfoil in an oscillating stream. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1372**, 38 pp. (9 plates) (1947).

Explicit formulas are derived for the velocity potential, lift force, moment and thrust on a two-dimensional flat airfoil in an oscillating stream varying its angle of attack sinusoidally. The derivations make the usual assumptions of incompressible potential flow, small oscillations and a trailing infinite plane vortex sheet. The lift and moment expressions are obtained by an extension of Theodorsen's method [Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **496** (1935)], and the thrust by application of thin airfoil theory. The results are in agreement with those of Garrick and Küssner, who used different methods of derivation. Applications are given. *D. Gilbarg* (Bloomington, Ind.).

Garrick, I. E., and Rubinow, S. I. Theoretical study of air forces on an oscillating or steady thin wing in a supersonic main stream. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1383**, 39 pp. (5 plates) (1947).

This paper studies the supersonic flow past oscillating or steady thin wings of finite span. The linearized equation for the velocity potential is solved by the method of source-sink distributions, based on the elementary solution for a moving sound source. Two régimes are distinguished in the boundary problem: (1) the "purely supersonic," (2) the "mixed supersonic," according to the parts of the fluid influenced by the airfoil, the former corresponding to independence between the upper and lower airfoil surfaces, the latter to an interaction. Both cases may occur simultaneously on a given wing. The source-sink method of this paper does not handle the mixed case. In the purely supersonic case it gives an expression for the potential which is applied to a number of examples. For wings of infinite span, the results reduce to the Ackeret theory for steady wings, and to previous results of the authors for nonstationary wings. *D. Gilbarg* (Bloomington, Ind.).

***Reissner, Hans. Blade systems of circular arrangement in steady, compressible flow.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 307–327. Interscience Publishers, Inc., New York, 1948. $5.50.

The problem discussed is, if the performance of a rotating or fixed blade system with circular symmetry and a certain characteristic of the flow is specified, to find the shape of the blading. The author first solves the problem for an infinite number of blades. For this, he generalizes the original idea of H. Lorenz [Neue Theorie und Berechnung der Kreiselräder . . . , Oldenbourg, München-Berlin, 1911] of replacing the blades by a continuous axially symmetric force field. The prescribed gradual change of pressure along stream lines and the prescribed main component of the velocity vector function determine the flow pattern. The continuity equation, the change of state condition, and the energy equation furnish the remaining two velocity components. The force field can then be calculated from the dynamic equations. Finally, the blade shapes are determined by integrating the differential equations for the stream lines.

When the number of blades is finite, the flow between the blades is no longer uniform. To calculate this effect, the author postulates one streamline surface, intermediate in each sector space between two blades, as frozen in the form which is determined by the infinite blade theory. The corrections for the asymmetry in this space are given by a scheme replacing the force field of the axially symmetric flow by the inertia forces of the added asymmetric accelerations of a corrected flow, which is taken as a power series in the angular distances from the frozen streamline surface. *H. S. Tsien* (Cambridge, Mass.).

Cap, Ferdinand. Über eine Erweiterung der Strömungs- und der Kontinuitätsgleichung der instationären Gasdynamik für den Fall des Vorhandenseins von Gasquellen und des Mitgerissenwerdens fester oder flüssiger Partikel. Acta Physica Austriaca **1**, 89–97 (1947).

The author first considers the alterations which need to be made in the continuity equation and the dynamic equation for one-dimensional nonsteady flow of nonviscous gas if there are (1) distributed sources of gas, (2) liquid or solid particles or (3) both sources of gas and liquid or solid particles in the gas. Then he shows how these equations can be solved by the method of characteristics. [The author's definition of a density for the liquid or solid particles seems to be questionable. Since the energy equation and the equation of states are not considered by the author, the solution obtained by the author, if correct, cannot be complete.] *H. S. Tsien* (Cambridge, Mass.).

Kên, Chin Shih. Remarques sur l'intégration approchée des equations du mouvement continu d'un fluide compressible. C. R. Acad. Sci. Paris **225**, 718–720 (1947).

The author states that he investigates the two compressible fluids under the assumption of the relations

$$\text{(a)}\qquad p=p_0-A/\rho,\quad q^2=A/\rho^2+B;$$

$$\text{(b)}\qquad p=p_0+A\int\frac{\rho^2 d\rho}{(\rho^2-B)^2},\quad q^2=\frac{A}{\rho^2-B},$$

where p is the pressure and ρ is the density, in order to determine which of the two fluids is a better approximation to the "real" compressible fluid both for subsonic and super-

sonic velocities. This work is a sequel to the work of J. Pérès [same C. R. 219, 501–504 (1944); these Rev. 7, 342].
A. Gelbart (Princeton, N. J.).

von Karman, Theodore. The similarity law of transonic flow. J. Math. Phys. Mass. Inst. Tech. 26, 182–190 (1947).

The problem of studying two-dimensional isentropic compressible flow in the transonic region is approached by representing the speed of flow as the critical speed of sound plus a small perturbation speed. For approximately sonic flows, it is shown that flow patterns around aerodynamic shapes of small thickness ratio τ are similar for equal values of the quantity $K=(1-M_1)/(\tau\Gamma)^{2/3}$, where M_1 is the Mach number of the flow at infinity and $\Gamma=(\nu+1)/2$ (ν is the ratio of specific heats). It is found, moreover, that for airfoils having similar thickness distributions the drag and lift coefficients vary as the 5/3 and 2/3 powers of τ, respectively. The conclusions are checked by comparison, in the case of slightly supersonic flow, with well-known exact solutions of supersonic flow problems. *E. N. Nilson.*

Guderley, G. Die Charakteristikenmethode für ebene und achsensymmetrische Überschallströmungen. Jahrbuch **1940** der Deutschen Luftfahrtforschung, I522–I535 (1940).

Die Busemannsche Methode zur Ermittlung ebener Potentialströmungen wird auf Grund einiger Überlegungen physikalischer Natur dahin erweitert, dass man alle Überschallströmungen, die nur von zwei Lagekoordinaten abhängen und für die der Wärmeinhalt des Kesselzustandes auf allen Stromlinien derselbe ist, grundsätzlich behandeln kann. Das Verfahren wird an einigen Beispielen erläutert; u.a. wird auch angegeben, welche Form man einer Düse, die einen vorgegebenen rotationssymmetrischen Überschallstrahl gleichrichtet, zu geben hat. *Author's summary.*

Tetervin, Neal. Boundary-layer momentum equations for three-dimensional flow. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1479**, 27 pp. (1947).

The author derives the momentum integral equations for three-dimensional boundary layer flow of a compressible fluid. The equations are the generalization of von Kármán's integral equation to the case where the outer flow depends on two coordinates. The reduction of the equations to the form recently given by Prandtl for incompressible flow is shown as well as a comparison with results obtained by Kehl for the flow in divergent and convergent channels.
H. W. Liepmann (Pasadena, Calif.).

Villey, Jean. La tuyère de Laval non isentropique à gaz parfait. J. Phys. Radium (8) **8**, 105–110 (1947).

The author adequately summarizes his paper as follows. Les équations aux dérivées logarithmiques de l'écoulement linéaire non isentropique des gaz parfaits permettent d'étudier, au cours de l'amorçage de la tuyère de Laval, les déplacements des sections où u, v, T et p passent par leur maximum ou leur minimum. Lorsque la vitesse d'écoulement augmente, ces quatre sections viennent se confondre en une seule, qui devient la section sonique. Cette section sonique n'est située au col que si l'apport de chaleur q et l'énergie décoordonnée w satisfont la relation $q/w=-\gamma/(\gamma-1)$. Cette condition est satisfaite en particulier par l'isentropie adiabatique ($q=w=0$), mais ne l'est pas dans le cas de l'isentropie par compensation ($q=-w$).
C. C. Torrance (Annapolis, Md.).

Brinkley, Stuart R., Jr., and Kirkwood, John G. Theory of the propagation of shock waves. Physical Rev. (2) **71**, 606–611 (1947).

The authors set out to determine the pressure rise across an advancing detonation wave as a function of its distance from the explosive source. The discussion is given in such a way as to apply to plane, cylindrical and spherical shocks. The two hydrodynamical equations for velocity and pressure in terms of the Lagrange coordinate ("particle coordinate") R and the time, and the Hugoniot relations across the moving shock are made to yield three equations in $\partial p/\partial R$, $\partial p/\partial t$, $\partial u/\partial R$, $\partial u/\partial t$, where here p and u refer to the abrupt pressure rise and the particle velocity just behind the shock.

An analysis is then given of the energy of the shock wave (i.e., the work it will do on the static fluid exterior to it) in which account is taken of the energy it leaves behind it, as a result of the entropy increment across the shock. The details in this section are somewhat difficult to follow, but the authors derive an expression relating shock wave energy to position. This expression involves a knowledge of the solution of the hydrodynamical equations. To avoid this difficulty, an empirically justified formula for the energy-time integrand is used, and the energy equation gives a fourth relation between the derivatives listed above. From these four relations two ordinary differential equations for the pressure excess p and the shock wave energy D (in terms of R) can be formulated. The coefficients, except in so far as they involve D explicitly, are expressed as functions of p by means of the Hugoniot relations and the (arbitrary) equation of state of the fluid. The solution must then proceed numerically. However, for the case of small pressure excess, the authors give analytic solutions which show that as $R\rightarrow\infty$, p vanishes (a) like $(R \ln R)^{-1}$ for spherical waves, (b) like $R^{-\frac{1}{2}}$ for cylindrical waves, (c) like $R^{-\frac{1}{2}}$ for plane waves. The evaluation of the two constants of integration involves a knowledge of the energy and the thermodynamic properties of the explosion products. *D. P. Ling.*

Hantzsche, W., und Wendt, H. Zum Verdichtungsstoss bei Zylinder- und Kugelwellen. Jahrbuch **1940** der Deutschen Luftfahrtforschung, I536–I538 (1940).

The problem solved is the condition for the appearance of shock in the propagation of cylindrical or spherical waves of finite amplitude from an axis or center. By using the differential equations of motion together with conditions at the wave front, the authors found that, if K is the gradient of velocity at the front when the front is at a distance d from the axis or the center, then for the appearance of shock, $-K\geqq a_0d^{-1}/(\kappa+1)$ for cylindrical waves, $-K\geqq 2a_0d^{-1}/(\kappa+1)$ for spherical waves, where a_0 is the velocity of sound at rest and κ is the ratio of specific heats. The result for a spherical wave agrees with that of Burton [Philos. Mag. (5) 33, 317–333 (1893)]. *H. S. Tsien.*

Ferrari, C. Sulla determinazione del profilo "ottimo" per le pale dei compressori assiali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 576–586 (1947).

The "optimal" form of the blade profile of an axial compressor is investigated. The author considers as optimal a profile which, for a given lift coefficient, has as uniform as possible a distribution of lift force along each profile. The solution of the problem is an approximate one, based upon certain assumptions valid only for thin profiles, and is given by aid of conformal mapping. As final result a definite profile shape is suggested. *P. Neményi* (Washington, D. C.).

Sbrana, Francesco. Ancora sul moto di un solido ellissoidico omogeneo immerso in un liquido. Rend. Circ. Mat. Palermo 62, 359–362 (1941).

Rossby, C.-G. Notes on the distribution of energy and frequency in surface waves. J. Marine Research 6, 93–103 (1947).

The author discusses the distribution of energy in the spectrum of surface waves in water which arise from a point source. The amount of energy $E_\lambda d\lambda$ contained between the wave lengths λ and $\lambda+d\lambda$ is shown to be independent of the time. The energy needed to establish the wave energy E_λ for larger and larger values of λ can be interpreted as the result of a rapid transport of potential energy from the point source outward, a small portion of this energy being converted into wave energy. A special wave motion of the water is studied in a second part of the paper in which a well-defined progressing wave front, within which the wave period increases with the time, moves into water at rest.

J. J. Stoker (New York, N. Y.).

Starr, Victor P. A momentum integral for surface waves in deep water. J. Marine Research 6, 126–135 (1947).

A very general relation between the kinetic energy and the horizontal momentum of surface waves (of any amplitude) is derived. The formula is then used to estimate the growth in amplitude of waves due to the tangential wind stress arising from a steady wind, with the result that waves of much greater wave length and period are obtained than are observed in the oceans. This discrepancy is explained by the author as being due to the fact that all of the momentum imparted to the water by the wind was assumed to be converted into wave motion while actually a large fraction of the momentum is likely to be used to produce drift currents in the ocean. *J. J. Stoker* (New York, N. Y.).

García, Godofredo. General cardinal equations of motion of viscous fluids. Actas Acad. Ci. Lima 10, 25–46 (1947). (Spanish)

García, Godofredo. Eulerian wind. Case of a compressible fluid. Actas Acad. Ci. Lima 10, 47–57 (1947). (Spanish)

In the first part the author considers various vector forms of the equations of motion of a general fluid. He then considers the special case of a geostrophic wind, familiar in meteorology, but with viscosity taken into account; he shows that the differential equations can be integrated under the assumption that the velocity is small. In the second paper he considers the case of a compressible "Eulerian" wind and derives a partial differential equation for a function related to a velocity potential. *W. Kaplan.*

✶Queney, Paul. Theory of perturbations in stratified currents with applications to air flow over mountain barriers. Publ. Dept. Meteorol. Univ. Chicago. Misc. Rep. no. 23. University of Chicago Press, Chicago, Ill., 1947. 81 pp. $1.50.

The author linearizes the hydrodynamic equations in the customary manner by assuming that small perturbations are superimposed on an undisturbed current. This undisturbed current satisfies the hydrodynamic equations, and the perturbations are assumed to be sufficiently small so that expressions of the second and higher order in the perturbation terms may be neglected. Special consideration is given to harmonic waves under the influence of gravity which are of particular interest in meteorology. These waves are mainly influenced by two factors, stability of the atmospheric stratification and Coriolis force. The effects of eddy viscosity and thermal conductivity are considered, and it is shown that these effects may be appreciable in the upper troposphere. Radiative conductivity is shown to be important for periods of one day or more. Application of the theory to the deformation by a mountain range of constant cross section shows that a system of stationary waves is set up which is more strongly developed downstream than upstream, except in the immediate vicinity of the obstacle. The displacement has a horizontal as well as a vertical component. The wave length has a minimum in the vertical direction, a maximum in the horizontal direction. The minimum value depends on the stability and the current velocity, the maximum value on the Coriolis parameter and the current velocity. The former is of the order of a few kilometers, the latter of the order of several hundred kilometers. The amplitude decreases first upwards, then it increases again, while in the horizontal direction it decreases and is inversely proportional to the square root of the distance.

B. Haurwitz (New York, N. Y.).

Prandtl, L. Zur Berechnung des Wetterablaufs. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. 1946, 102–105 (1946).

The five fundamental equations of meteorology are discussed from the point of view of their numerical integration. They are combined in an equation for the pressure change. This shows again that geostrophic wind cannot be used for the determination of pressure change. [The equation is essentially a combination of the Bjerknes tendency equation and the vorticity equation.] The orders of magnitude of the terms of the fundamental equations are estimated. [Some of these estimates do not agree with meteorological experience.] An order of numerical integration is suggested.

H. A. Panofsky (New York, N. Y.).

Elasticity, Plasticity

Hill, R., Lee, E. H., and Tupper, S. J. The theory of combined plastic and elastic deformation with particular reference to a thick tube under internal pressure. Proc. Roy. Soc. London. Ser. A. 191, 278–303 (1947).

The first five sections of this paper contain a general discussion of stress-strain relations of perfectly plastic materials, while the remaining five sections are concerned with the application to the problem mentioned in the title. The discussion of the stress-strain relations is based on a principle of maximum energy dissipation. For a given yield condition, this leads to the same stress-strain relations as Mises' theory of flow potential [Z. Angew. Math. Mech. 8, 161–185 (1928); see also the article by Reuss, ibid. 12, 15–24 (1932)]. As a matter of fact, both Mises and Reuss have formulated their theories in terms of stationary energy dissipation [loc. cit., pp. 162 and 15, respectively]. The analysis of the stress distribution in a thick-walled tube under internal pressure is based on the assumption of plane strain. A further simplification is obtained by replacing the circle which represents Mises' yield condition by a tangent; it is shown that in the case under consideration this approximation does not involve deviations from Mises' yield condition which exceed 3%. It is pointed out that neglect of the elastic strains in the plastic region causes grave errors in the axial stresses. *W. Prager* (Providence, R. I.).

Taylor, Geoffrey. A connexion between the criterion of yield and the strain ratio relationship in plastic solids. Proc. Roy. Soc. London. Ser. A. **191**, 441–446 (1947).

Stress-strain relations of perfectly plastic materials are discussed in the light of a principle of maximum energy dissipation [see the preceding review]. It is shown that the results of earlier experiments by Taylor and Quinney [Philos. Trans. Roy. Soc. London. Ser. A. **230**, 323–362 (1931)] agree fairly well with the prediction based on this principle. *W. Prager* (Providence, R. I.).

Lifšic, I. M., and Rozencveĭg, L. N. On the construction of the Green's tensor for the fundamental equations of the theory of elasticity in the case of an unrestricted elastic-anisotropic medium. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **17**, 783–791 (1947). (Russian)

Riz, P. M. Elastic constants in the non-linear theory of elasticity. Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 493–494 (1947). (Russian. English summary)

The note deals with the choice of elastic constants in a modification of Hooke's law suggested by F. D. Murnaghan, so that it will characterize accurately the dependence of stresses on strains in an isotropic body.

I. S. Sokolnikoff (Los Angeles, Calif.).

***Schüepp, H. Spannung und Spannungstensor.** Elemente der Math. Beihefte no. **1**, 24 pp. (1947).

Narodeckiĭ, M. Z. On the problem of Hertz on the contact of two cylinders. Doklady Akad. Nauk SSSR (N.S.) **56**, 463–466 (1947). (Russian)

The author applies the complex variable methods of solution of plane problems of elasticity to solve the problem of interior contact of two perfectly smooth circular cylinders C_1 and C_2. The cylinders are infinitely long and C_2 is interior to C_1. If the plane of the complex variable $z=x+iy$ is taken perpendicular to the axes of the cylinders, then their cross-sections S_1 and S_2 are circular regions of radii R_1 and R_2, $R_1>R_2$. The force P pressing C_2 toward C_1 is applied at an arbitrary point of the y-axis in the region S_2 ($|z|\leq R_2$) and is directed along the y-axis. It is known that the solution of the elastic problem is determined by a set of four analytic functions $\varphi_j(z)$ and $\psi_j(z)$, determined in the regions S_j ($j=1, 2$), where S_1 is the region exterior to the circle of radius R_1. These functions satisfy along the common arc of contact γ the boundary condition

$$(1)\qquad \varphi_j'(t)+\overline{\varphi_j'(t)}-\overline{t\varphi_j''(t)}-e^{-2i\theta}\overline{\psi_j'(t)}=N(t),$$

where $N(t)$ is the normal component of stress along γ (the cylinders are smooth) and $\theta=\arg z$. The function $N(t)$ is determined by the load P, and the elastic constants κ and μ are assumed to be the same for S_1 and S_2. The functions φ_j and ψ_j are connected with the displacement vector $u+iv$ by the formulas

$$2\mu(u_j+iv_j)=\kappa\varphi_j(z)-z\overline{\varphi_j'(z)}-\overline{\psi_j(z)}.$$

The continuity of displacements along γ, together with the boundary condition (1) on stresses, permit the author to calculate these functions and compute the maximum stresses (occurring in the middle of the arc of contact) for several configurations. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Mindlin, Ya. A. Boundary problems of the theory of elasticity in the case of a circle. Doklady Akad. Nauk SSSR (N.S.) **56**, 249–252 (1947). (Russian)

Generalizing his previous work [same Doklady (N.S.) **15**, 531–534 (1937); **16**, 15–18 (1937)] the author considers the problem of oscillation of a circle in its plane, when the initial values of displacement and velocity and the boundary values of displacement are arbitrarily assigned as Fourier series of θ, where (r, θ) are polar coordinates. The displacement vector is represented as a linear combination of partial derivatives of two functions $f_i(r, \theta, t)$, each satisfying a wave equation. The functions f_i are expressed as Fourier series of θ with coefficients put in form of Fourier integrals. The integrand functions of the latter are determined as solutions of integral equations of Schlömilch type, which the author considered in another paper [C. R. (Doklady) Akad. Nauk SSSR (N.S.) **56**, 141–144 (1947); these Rev. **9**, 94]. *I. Opatowski* (Ann Arbor, Mich.).

Carrier, G. F. On the buckling of elastic rings. J. Math. Phys. Mass. Inst. Tech. **26**, 94–103 (1947).

This paper discusses four problems involving the deformation of plane closed elastic rings (or arches) of small depth. The basic theory used corresponds to the elastica theory for straight beams, and is correspondingly nonlinear. The first problem considered is that of stability of a closed ring under loads which are such as to cause as sole deformation a shortening of the middle line of the ring (in the case of a circular ring, for example, this results when the load is a uniform normal pressure); the load at which instability occurs is then determined without too much difficulty by a linear eigenvalue problem of third order. This problem has been investigated by Biezeno and Koch for the special case of the circular ring [Nederl. Akad. Wetensch., Proc. **48**, 447–468 (1945); these Rev. **8**, 360].

The second problem treated is the problem of nonlinear buckling of the circular ring under normal pressure; numerical solutions are given for the lowest buckling "mode" for pressures well beyond the load at which instability begins; as in the analogous case of the elastica the basic differential equation can be solved in terms of elliptic functions, but it is necessary to solve a rather complicated set of transcendental equations in order to satisfy the periodicity conditions imposed by the fact that the ring is closed. The third problem treated is the stability of a ring which has been first subjected to forces causing a large deformation. Finally, the author formulates the problem of the dynamic stability of the circular ring under uniform pressure whose intensity varies periodically in the time. *J. J. Stoker.*

Reinitzhuber, F. Über die Stabilität gerader Stäbe mit linear veränderlicher Langskraft. Jahrbuch **1940** der Deutschen Luftfahrtforschung, I820–I824 (1940).

Es werden die Knicklasten für Stäbe mit linear veränderlicher Längskraft in drei Befestigungsfällen mit Hilfe des Verfahrens von Galerkin angenähert bestimmt.

Author's summary.

Müller, R., und Müller, W. Knickung und Knickbiegung von konischen Stäben. Jahrbuch **1941** der Deutschen Luftfahrtforschung, I502–I517 (1941).

Die Knicklast von konischen Stäben wird für die hauptsächlichsten Belastungsfälle berechnet und in Kurven über dem Trägheitsmomentenverhältnis dargestellt. Ferner wird das Knickbiegemoment des einseitig eingespannten konischen Stabes unter der Einwirkung einer Querkraft sowie eines Momentes ermittelt und ebenfalls durch Kurven veranschaulicht. *Author's summary.*

Seth, B. R. Bending of clamped rectilinear plates. Philos. Mag. (7) **38**, 292–297 (1947).

Using the known fact that a biharmonic function can be written in terms of two analytic complex functions, the author determines the deflection of clamped plates with rectilinear edges. By successive Schwartz-Christoffel and bilinear conformal mappings the region is mapped onto the interior of a circle and the mapping function is expanded in a power series. The two analytic functions mentioned above are also expressed in power series (in the circle plane) and an infinite set of algebraic equations is obtained for the unknown coefficients. The author carries out the expansions for equilateral plates and indicates that the results agree with the known solution for the square plate.

G. F. Carrier (Providence, R. I.).

Volkersen, O. Beitrag zur Berechnung rechteckiger versteifter Membranen. Jahrbuch **1940** der Deutschen Luftfahrtforschung, I873–I878 (1940).

Die Formeln für die unversteifte rechteckige Membran werden mit den Ansätzen von A. und L. Föppl abgeleitet. Durch Gleichsetzen der Biegungspfeile für die unversteifte Membran und die Versteifungsprofile erhält man eine Überschlagsformel für die Berechnung von Membranen, die in Richtung einer Achse versteift sind. Die Gültigkeit dieser Formel wird durch einige Stichprobenversuche nachgewiesen. *Author's summary.*

Vlasov, V. Z. Momentless theory of thin shells of revolution. Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 397–408 (1947). (Russian. English summary)

The author makes use of the equations of the momentless theory of shells to solve several problems involving elliptical, spherical, and parabolic cupolas subjected to prescribed loads. It is shown that the equations of momentless theory are reducible to the Cauchy-Riemann equations if the shells are quadric surfaces with positive Gaussian curvature.

I. S. Sokolnikoff (Los Angeles, Calif.).

Goldenweiser, A. L. Approximate calculation of thin shells of zero Gauss curvature. Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 409–422 (1947). (Russian. English summary)

The main object of this paper is a qualitative analysis of stressed states in thin elastic shells with developable middle surfaces. The paper also contains an outline of the methods of approximate calculation of stresses. The shell is covered by a net of lines of curvature α, β (the α-lines are of zero curvature) so that the first fundamental form for the surface is of the type $ds^2=d\alpha^2+B^2d\beta^2$. In this case Love's general equations of the shell theory are reducible to two differential equations for the stress functions t and m from which the forces, moments, and deformations can be computed by differentiation. These equations are:

$$\frac{\lambda^2}{B^2}\frac{\partial}{\partial\alpha}B^2\frac{\partial t}{\partial\alpha}-\frac{\lambda^2h^2}{3(1-\sigma^2)}N(m,\sigma)=0,$$
$$\frac{\lambda^3}{B^3}\frac{\partial}{\partial\alpha}B^2\frac{\partial m}{\partial\alpha}+\lambda^3N(t,-\sigma)=0,$$

where λ^2 and λ^3 are introduced to make the terms of these equations have the dimensions of t; for cylindrical shells $(\partial B/\partial\alpha=0)$,

$$N(F)=\frac{1}{B}\frac{\partial}{\partial\beta}\frac{B}{B}\frac{\partial}{\partial\beta}\frac{1}{B}\frac{\partial}{\partial\beta}\frac{1}{B}\frac{\partial F}{\partial\beta}+\frac{1}{BR}\frac{\partial}{\partial\beta}\frac{1}{B}\frac{\partial F}{\partial\beta},$$

R being the radius of curvature of the β-line; for noncylindrical shells $(\partial B/\partial\alpha\neq 0)$ $N(F)$, in addition to the terms given above, contains the term

$$-\frac{1}{B}\frac{\partial}{\partial\beta}\frac{R}{B}\left(\frac{\partial}{\partial\alpha}\frac{1}{B}\frac{\partial B}{\partial\alpha}\right)\frac{\partial F}{\partial\beta}.$$

For conical and cylindrical shells the system can be integrated approximately in the form of a series involving trigonometric and Bessel functions, provided certain restrictions on the lengths of the shells and on the generatrix angle are imposed.

Several results obtained in the author's two earlier papers [same journal **9**, 463–478 (1945); **10**, 387–396 (1946); these Rev. **7**, 351; **8**, 241], dealing with thin shells of zero Gaussian curvature which are so stressed that the state of stress can be decomposed into a momentless state and into a state produced by moments and boundary effects, appear as special cases in this more general treatment.

I. S. Sokolnikoff (Los Angeles, Calif.).

Polya, George. Sur la fréquence fondamentale des membranes vibrantes et la résistance élastique des tiges à la torsion. C. R. Acad. Sci. Paris **225**, 346–348 (1947).

Let D be a domain in the plane bounded by a single closed curve C, A the area of the domain, I its polar moment of inertia with respect to its center of gravity, Λ the fundamental frequency of the membrane stretched over C, and M the torsional resistance of an elastic cylinder with D as cross section. By Steiner symmetrization with respect to a line A is not changed, I and Λ are diminished, and M is increased. The result concerning Λ is due to the author and G. Szegö [Amer. J. Math. **67**, 1–32 (1945); these Rev. **6**, 227]; the results concerning I and M are announced without proof. An approximate formula for Λ, $\Lambda^2\sim 12\pi^2I/A^3$, is given and compared with another similar formula due to St. Venant for the quantity M. *J. J. Stoker.*

Fu, C. Y. On seismic rays and waves. I. Bull. Seismol. Soc. America **37**, 331–346 (1947).

The equations of motion of an elastico-viscous medium in Maxwell's sense are deduced and it is shown that the condition for separation of simple harmonic waves of divergence and curl is that the gradients of the elasticity and density be slight over the distance corresponding to a wave length. In such a medium the velocity function is shown to be in general a complex quantity in need of physical interpretation from other considerations. The author then sets up a corresponding wave front equation in which surfaces of equal amplitude are not necessarily parallel to those of like phase, and the propagation of the equiphase surfaces assumes Fermat's principle in a generalized form which gives Hamilton's equation of the characteristic function. The author shows by the method of Sommerfeld and Runge [Ann. Physik (4) **35**, 277–298 (1911)] that the condition for the existence of a true motion corresponding to the generalized ray equation is that the divergence of the gradient of the complex ray function be very much less than the square of the gradient itself. The ray theory then becomes a limiting case of the wave theory. Limiting the investigation to a harmonic motion with a real velocity the author arrives at a criterion for the validity of the ray method which is similar to de Broglie's [J. Phys. Radium (6) **7**, 321–337 (1926)] criterion in wave mechanics, that the frequency be sufficiently high. Using Epstein's method the author applies his results to several problems in pure and applied seismology and concludes that the ray method is usually valid. *J. B. Macelwane* (St. Louis, Mo.).

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